31B Notes

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1 Exponential Functions (7.1)

The following theorem pretty much summarizes section 7.1.

Theorem 1.1. *Rules for exponentials:*

(1) *Exponential Rules (Algebra):*

   (a) $a^x \cdot a^y = a^{x+y}$

   (b) $\frac{a^x}{a^y} = a^{x-y}$

   (c) $(a^x)^y = a^{xy}$

(2) *Derivatives of Exponentials:*

   (a) $f(x) = e^x \implies f'(x) = e^x$

   (b) $f(x) = a^x \implies f'(x) = a^x \ln(a)$

Example 1.2. *Simplify* $10^2(2^{-2} + 5^{-2})$

Solution. We want to use 1(a) above, but we can't since the bases aren't the same. So first write $(10)^2 = (2 \cdot 5)^2 = 2^2 \cdot 5^2$. Now

$$\begin{aligned}
10^2(2^{-2} + 5^{-2}) &= 2^2 \cdot 5^2(2^{-2} + 5^{-2}) \\
&= 2^2 \cdot 2^{-2} \cdot 5^2 + 2^2 \cdot 5^2 \cdot 5^{-2} \\
&= 2^{2+(-2)} \cdot 5^2 + 2^2 \cdot 5^{2+(-2)} \quad \text{(by 1(a) above)} \\
&= 5^2 + 2^2 \\
&= \boxed{29}
\end{aligned}$$

Example 1.3. *Simplify* $(9^{-1/6})^3$.

Solution. To simplify this, use 1(c) above:

$$(9^{-1/6})^3 = 9^{3 \cdot -1/6} = 9^{-1/2} = \frac{1}{9^{1/2}} = \boxed{\frac{1}{3}}$$

Example 1.4. *Solve* $2^{x+1} = 4$.

Solution. You know the answer should be $x = 1$. But we want to try and get this with a method that works more generally. To solve this, we want a common base on both sides. In this example, it will be 2, as $4 = 2^2$. So we want to solve $2^{x+1} = 2^2$. Since we now have the desired common base, we set the exponents equal (this works because exponentials are 1-1 functions). So $x + 1 = 2$, which means $\boxed{x = 1}$.

Example 1.5. *Solve* $5^x = 25^{1-x}$.

Solution. To solve this, we want to get the same base, so write $25 = 5^2$. Therefore

$$5^x = 25^{1-x} \implies 5^x = (5^2)^{1-x} \implies 5^x = 5^{2(1-x)}.$$

Now set the exponents equal:

$$x = 2(1-x) \implies x = 2 - 2x \implies 3x = 2 \implies x = \boxed{\frac{2}{3}}.$$

Example 1.6. *What is the tangent line to $f(x) = e^{\sin(x)}$ at $x = 0$?*

Solution. First, we want the y-coordinate: $f(0) = e^{\sin(0)} = e^0 = 1$. Next, we need the derivative for the slope:

$$f'(x) = e^{\sin(x)} \cdot \cos(x)$$

by the chain rule. Plugging in $x = 0$: $f'(0) = e^{\sin(0)} \cdot \cos(0) = e^0 \cdot 1 = 1$. Therefore, the tangent line is $\boxed{y - 1 = 1(x - 0)}$.

Example 1.7. *Find the derivative of $f(x) = xe^{-2x+3}$.*

Solution. We need product rule here (and chain rule too):

$$f'(x) = x \cdot (e^{-2x+3})' + e^{-2x+3} \cdot (x)' = \boxed{x \cdot e^{-2x+3} \cdot (-2) + e^{-2x+3}}.$$

Example 1.8. *Find the derivative of $f(x) = \frac{e^{x+1}}{(2x+1)^3}$.*

Solution. Quotient rule and chain rule:

$$f'(x) = \boxed{\frac{(2x+1)^3 \cdot e^{x+1} - e^{x+1} \cdot 3(2x+1)^2 \cdot 2}{(2x+1)^6}}.$$

Example 1.9. *Show $f(x) = x + e^{2x}$ is always increasing.*

Solution. Observe that $f'(x) = 1 + 2e^{2x}$. Since $e^{2x} > 0$ (as e to any power is > 0), $2e^{2x} > 0$, and finally if we add 1 it is more positive, so $1 + 2e^{2x} > 0$. Since $f'(x) > 0$ for all x, $f(x)$ is always increasing.

Example 1.10. *Evaluate $\int e^{2x+3} dx$.*

Solution. If you're comfortable just adjusting for constants without doing u-sub, fine. If not, do $u = 2x + 3$, so $du = 2dx$. We don't have the 2, so move it over to get $\frac{1}{2} du = dx$. This gives

$$\int e^{2x+3} dx = \frac{1}{2} \int e^u du = \frac{1}{2} e^u + C = \boxed{\frac{1}{2} e^{2x+3} + C}.$$

Example 1.11. *Evaluate $\int \sec(x) \tan(x) e^{\sec(x)} dx$.*

Solution. This is another u-sub, this time with $u = \sec(x)$, and $du = \sec(x)\tan(x)$. So

$$\int \sec(x) \tan(x) e^{\sec(x)} = \int e^u du = e^u + C = \boxed{e^{\sec(x)} + C}.$$

Example 1.12. *Evaluate $\int x\sqrt{e^{x^2}} dx$.*

Solution. The trick here is to notice that

$$\sqrt{e^{x^2}} = (e^{x^2})^{1/2} = e^{\frac{x^2}{2}}.$$

So the integral is just $\int xe^{x^2/2} dx$. To integrate, let $u = x^2/2$, $du = xdx$, and again the integral becomes just $\int e^u du = e^u + C = \boxed{e^{x^2/2} + C}$.

2 Inverse Functions and their derivatives (7.2)

This section was all about inverse functions.

Definition 2.1 (Inverse Function)**.** *If $f(x)$ is a function, then another function $g(x)$ is the* inverse *of f if $f(g(x)) = g(f(x)) = x$.*

Example 2.2. *Show directly from the definition that $f(x) = 3x - 5$ has inverse $g(x) = \frac{x+5}{3}$.*

Solution. We need to compute both $f(g(x))$ and $g(f(x))$. Just evaluate:

$$f(g(x)) = f\left(\frac{x+5}{3}\right) = 3\left(\frac{x+5}{3}\right) - 5 = x + 5 - 5 = x,$$

and

$$g(f(x)) = g(3x-5) = \frac{(3x-5)+5}{3} = \frac{3x}{3} = x.$$

Therefore g is the inverse of f.

Example 2.3. *If $f(x) = x^2 - 1$ for $x \le 0$, show from the definition that $g(x) = -\sqrt{x+1}$ is the inverse of $f(x)$.*

Solution. Again, we need $f(g(x))$ and $g(f(x))$:

$$f(g(x)) = f(-\sqrt{x+1}) = (-\sqrt{x+1})^2 - 1 = x + 1 - 1 = x,$$

and

$$g(f(x)) = g(x^2-1) = -\sqrt{(x^2-1)+1} = -\sqrt{x^2} = -|x|.$$

Now since the domain of $f(x)$ was $x \le 0$ and $|x| = -x$ on this region, we get that $-|x| = -(-x) = x$, which is what we needed. Therefore g is the inverse of f.

Functions need to be 1-1 to be invertible:

Definition 2.4. *A function $f(x)$ is* one-to-one *if $f(a) = f(b) \implies a = b$*

Remark. Graphically, this is saying the graph of $y = f(x)$ passes the horizontal line test. This means that every horizontal line crosses the graph of $y = f(x)$ at most once.

Example 2.5. *Show that the function $f(x) = 3x + 1$ is 1-1 algebraically.*

Solution. We want to show that if $f(a) = f(b)$, then $a = b$. If $f(a) = f(b)$, then $3a + 1 = 3b + 1$. Subtracting 1 gives $3a = 3b$, or $a = b$, which is what we needed. Therefore $f(x)$ is 1-1.

Remark. You could also have seen this clearly from the graph, since it is just a line, and so it will pass the horizontal line test.

Example 2.6. *Show that the function $f(x) = \frac{x}{x+1}$ is 1-1 algebraically.*

Solution. Again, start with $f(a) = f(b)$:

$$\begin{aligned} f(a) = f(b) \quad &\implies \quad \frac{a}{a+1} = \frac{b}{b+1} \\ &\implies \quad a(b+1) = b(a+1) \quad \text{(cross multiply)} \\ &\implies \quad ab + a = ab + b \\ &\implies \quad a = b. \end{aligned}$$

Therefore $f(x)$ is 1-1.

The last key point for this part of the chapter:

Theorem 2.7. *If a continuous function is always increasing or always decreasing, then it is 1-1.*

Example 2.8. *Show that $f(x) = x + e^{2x}$ is 1-1.*

Solution. Your first guess might be to try the definition: $f(a) = f(b) \implies a = b$. You quickly find that it is very difficult to get to $a = b$. So you try something else. In this case, looking at Example 1.9, we know $f(x)$ is always increasing. Therefore $f(x)$ is 1-1.

So to summarize:

Method 2.9. *To show a function $f(x)$ is invertible, you can:*

(1) *Find the inverse directly: you do this by switching x and y and solving for y if possible.*

(2) *Show $f(x)$ is 1-1 by either:*

  (a) *Graphing it and showing it passes the horizontal line test*
  (b) *Using the definition: $f(a) = f(b) \implies a = b$*
  (c) *Show it is always increasing or always decreasing (and you should say it is continuous).*

Finally, we need to be able to differentiate inverse functions:

Theorem 2.10 (Derivative of Inverses)**.** *Suppose $g(x)$ is the inverse of $f(x)$. If $x = b$ is in the domain of $g(x)$ (hence, in the range of $f(x)$), then*

$$g'(b) = \frac{1}{f'(g(b))}.$$

The theorem really looks more complicated than it should be. Here's going to be the general approach:

Method 2.11 (Derivative of Inverses)**.** *If you want the derivative of $g(x)$, where $g(x) = f^{-1}(x)$:*

(1) *Find $f'(x)$ first. If a point is given, do step 2. If a point is not given, go to step 3.*

(2) *Point given:*

  (a) *If a point $x = b$ is given: Find $f^{-1}(b)$ by solving the equation $f(x) = b$.*
  (b) *Plug the value from (a) into f'.*
  (c) *Take the reciprocal.*

(3) *Point not given:*

  (a) *Find $f^{-1}(x)$.*
  (b) *Plug the function from (a) into f'.*
  (c) *Take the reciprocal.*

Example 2.12. *If $f(x) = \frac{x^3}{4}$, find $\frac{d}{dx} f^{-1}(x)\Big|_{x=16}$, the tangent line to $f^{-1}(x)$ at $x = 16$, and $\frac{d}{dx} f^{-1}(x)$ using the previous theorem.*

Solution. In the formula we need f', so $f'(x) = \frac{3x^2}{4}$. Notice that you don't plug in the given point into f'. You need to first figure out f^{-1} of your point, in this case, $f^{-1}(16)$. You do this by setting $f(x) = 16$ and seeing which x gave you 16 as the output:

$$f(x) = 16 \implies \frac{x^3}{4} = 16 \implies x^3 = 64 \implies x = 4.$$

So $x = 4$ is what we plug into $f'(x)$ to get

$$f'(4) = \frac{3 \cdot 4^2}{4} = 12.$$

So

$$\frac{d}{dx} f^{-1}(x)\Big|_{x=16} = \boxed{\frac{1}{12}}.$$

For the tangent line, we just found the slope. The point is $(16, f^{-1}(16)) = (16, 4)$, so the line is

$$\boxed{y - 4 = \frac{1}{12}(x - 16)}.$$

Finally, to find the derivative of the inverse, use step 3 in the method outlined above. We know we need $f^{-1}(x)$. Switch x and y and solve for y:

$$x = \frac{y^3}{4} \implies y = (4x)^{1/3}.$$

Now plug this into f' to get $f'(f^{-1}(x)) = \frac{3((4x)^{1/3})^2}{4}$. Finally, take the reciprocal to get

$$\frac{d}{dx} f^{-1}(x) = \boxed{\frac{4}{3(4x)^{2/3}}}.$$

Example 2.13. *If $f(x) = e^{\tan(x)}$, $-\pi/2 < x < \pi/2$, find $\frac{d}{dx} f^{-1}(x)\Big|_{x=e^{\sqrt{3}}}$.*

Solution. Again, first find $f'(x)$: $f'(x) = \sec^2(x)e^{\tan(x)}$. Next, we need $f^{-1}(e^{\sqrt{3}})$:

$$e^{\sqrt{3}} = e^{\tan(x)} \implies \sqrt{3} = \tan(x) \implies x = \pi/3.$$

Since $f'(\pi/3) = \sec^2(\pi/3)e^{\tan(\pi/3)} = 4e^{\sqrt{3}}$, we get

$$\frac{d}{dx} f^{-1}(x)\Big|_{x=e^{\sqrt{3}}} = \boxed{\frac{1}{4e^{\sqrt{3}}}}.$$

3 Logarithms and their derivatives (7.3)

Here's the main part of the section:

Theorem 3.1. *Rules for logarithms:*

(1) Algebra:

  (a) $\log_a(xy) = \log_a(x) + \log_a(y)$
  (b) $\log_a(x/y) = \log_a(x) - \log_a(y)$
  (c) $\log_a(x^y) = y \log_a(x)$

(2) Calculus rules:

  (a) If $f(x) = \log_a(x)$, then $f'(x) = \frac{1}{\ln(a)} \cdot \frac{1}{x}$.
  (b) If $f(x) = \frac{1}{x}$, then $\int f(x)dx = \ln|x| + C$.

Example 3.2. *What is $\log_2(1/4)$?*

Solution. Just think of log as an exponent: If $x = \log_2(1/4)$, then $2^x = \frac{1}{4} = 2^{-2}$, so $x = \boxed{-2}$.

Example 3.3. *If $\log_a(x) = 3$ and $\log_a(y) = -1$, find $\log_a\left(\frac{x^2}{y^3}\right)$.*

Solution. Log rules:

$$\log_a\left(\frac{x^2}{y^3}\right) = \log_a(x^2) - \log_a(y^3) = 2\log_a(x) - 3\log_a(y) = 2(3) - 3(-1) = \boxed{9}.$$

Example 3.4. *Solve the equation $e^{2x+3} = 5$.*

Solution. The point is $\ln(e^u) = u$ and $e^{\ln(u)} = u$ as e^x and $\ln(x)$ are inverses. So in this case, to isolate x, take logs of both sides:

$$\ln(e^{2x+3}) = \ln(5) \implies 2x + 3 = \ln(5).$$

Therefore $x = \boxed{\dfrac{\ln(5) - 3}{2}}$.

Example 3.5. *Solve* $e^{2\ln(x)+3} = 5$.

Solution. Notice you can't use the $e^{\ln(u)} = u$ relation yet because of the 2 and 3 in the exponent. But recall that $e^{2\ln(x)+3} = e^{2\ln(x)} \cdot e^3$. Also, by log rules, $2\ln(x) = \ln(x^2)$. Putting these together, the equation becomes

$$e^{\ln(x^2)} \cdot e^3 = 5 \implies x^2 = \frac{5}{e^3} \implies x = \pm\sqrt{\frac{5}{e^3}}.$$

However, we can't have $x < 0$ since the domain of $\ln(x)$ is $x > 0$. Hence, $x = \boxed{\sqrt{\dfrac{5}{e^3}}}$.

Example 3.6. *Find the equation of the tangent line to* $f(x) = \ln(x^3 + 1)$ *at* $x = 0$.

Solution. First, we note that $f'(x) = \frac{1}{x^3+1} \cdot 3x^2$ by the chain rule, so $f'(0) = 0$. Also, $f(0) = \ln(1) = 0$. Therefore the line is $y - 0 = 0(x - 0)$, or $\boxed{y = 0}$.

Example 3.7. *Find the derivative of* $y = \ln\left(\frac{(x+1)\sin(x)}{e^x+1}\right)$.

Solution. Before using chain rule, simplify the expression using log rules:

$$\begin{aligned} \ln\left(\frac{(x+1)\sin(x)}{e^x+1}\right) &= \ln((x+1)\sin(x)) - \ln(e^x+1) \\ &= \ln(x+1) + \ln(\sin(x)) - \ln(e^x+1). \end{aligned}$$

So $y = \ln(x+1) + \ln(\sin(x)) - \ln(e^x+1)$, so when we take the derivative, we get

$$y' = \boxed{\frac{1}{x+1} + \frac{1}{\sin(x)} \cdot \cos(x) - \frac{1}{e^x+1} \cdot e^x}.$$

Example 3.8. *Evaluate* $\int \frac{dx}{3x+5}$.

Solution. You can probably jump to the answer, but in case you have trouble thinking about adjusting for constants, do a u-sub: $u = 3x + 5$, $du = 3dx$, so $dx = \frac{1}{3}du$. The integral then becomes

$$\frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|u| + C = \boxed{\frac{1}{3}\ln|3x+5| + C}.$$

Example 3.9. *Evaluate* $\int \frac{dx}{(2x-1)\ln(4x-2)}$.

Solution. This is another u-sub: $u = \ln(4x-2)$, $du = \frac{1}{4x-2} \cdot 4dx$. First, write this as $\frac{1}{4}du = \frac{1}{4x-2}dx$. Next, you notice that in the integral, you only have $2x - 1$, not $4x - 2$, so if you multiply both sides by 2, you get $\frac{1}{2}du = \frac{1}{2x-1}dx$. So the integral is

$$\frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \boxed{\frac{1}{2}\ln|4x-2| + C}.$$

Example 3.10. *Integrate* $\tan(x)$.

Solution. Write $\tan(x) = \frac{\sin(x)}{\cos(x)}$, so we want $\int \frac{\sin(x)}{\cos(x)} dx$. Let $u = \cos(x)$, $du = -\sin(x)dx$, so $-du = \sin(x)dx$. We have

$$-\int \frac{du}{u} = -\ln|u| + C = \boxed{-\ln|\cos(x)| + C}.$$

Logarthmic Differentiation: We use log differentiation in two main cases: when there are x's in the base and the exponent of a function, or when taking logs and simplifying would make the derivative easier, as in the case of

$$y = \frac{e^{x^2}\sqrt{x+1} \cdot 2^x}{\sec^2(x)\sin^2(x)\sqrt[4]{x^2+1}}.$$

Example 3.11. *Find the derivative of* $y = (2x+1)^{\sin(x)}$.

Solution. We need to use log differentiation because we have x's in the base and the exponent. So take logs of both sides:

$$\ln(y) = \ln((2x+1)^{\sin(x)}) = \sin(x)\ln(2x+1).$$

Taking derivatives, and remembering implicit differentiation on the left:

$$\frac{1}{y} \cdot y' = \sin(x) \cdot \frac{2}{2x+1} + \ln(2x+1)\cos(x),$$

so

$$y' = y\left(\sin(x) \cdot \frac{2}{2x+1} + \ln(2x+1)\cos(x)\right) = \boxed{(2x+1)^{\sin(x)}\left(\sin(x) \cdot \frac{2}{2x+1} + \ln(2x+1)\cos(x)\right)}.$$

Remark. You CANNOT use the $a^x \mapsto a^x \ln(a)$ rule here; this only works for a a positive **constant**. Make sure you remember this.

Example 3.12. *If $f(x)$ is a differentiable function, find the derivative of $x^{f(x)}$.*

Solution. Again, write $y = x^{f(x)}$ and take logs:

$$\ln(y) = \ln(x^{f(x)}) = f(x)\ln(x).$$

Taking derivatives:

$$\frac{1}{y}y' = \frac{f(x)}{x} + \ln(x)f'(x),$$

so

$$y' = y\left(\frac{f(x)}{x} + \ln(x)f'(x)\right) = \boxed{x^{f(x)}\left(\frac{f(x)}{x} + \ln(x)f'(x)\right)}.$$

Example 3.13. *Find the derivative of* $y = \frac{(x+1)\sin(x)}{e^x+1}$.

Solution. Now, you could differentiate this using quotient rule, and then product rule on top, and so on. The alternative would be to realize you have a bunch of products and quotients, and logs help you break these up. So take ln of both sides:

$$\ln(y) = \ln\left(\frac{(x+1)\sin(x)}{e^x+1}\right).$$

We took the derivative of the right side in example 3.7. The left will become $\frac{1}{y} \cdot y'$, so altogther we get

$$\frac{1}{y} \cdot y' = \frac{1}{x+1} + \frac{1}{\sin(x)} \cdot \cos(x) - \frac{1}{e^x+1} \cdot e^x,$$

so

$$y' = y\left(\frac{1}{x+1} + \frac{1}{\sin(x)} \cdot \cos(x) - \frac{1}{e^x+1} \cdot e^x\right) = \boxed{\frac{(x+1)\sin(x)}{e^x+1}\left(\frac{1}{x+1} + \frac{1}{\sin(x)} \cdot \cos(x) - \frac{1}{e^x+1} \cdot e^x\right)}.$$

4 Exponential Growth and Decay (7.4)

Summary 4.1. *(1) Exponential Growth:*

- *(a) Differential Equation:* $y' = ky$
- *(b) Equation:* $y = y_0 e^{kt}$
- *(c) Doubling time:* $\frac{\ln(2)}{k}$

(2) Exponential Decay

- *(a) Differential Equation:* $y' = -ky$
- *(b) Equation:* $y = y_0 e^{-kt}$
- *(c) Half-life:* $\frac{\ln(2)}{k}$

Remark. Notice that both the half-life and the doubling time are the same. Also note that all the k's you see in the summary are assumed to be positive.

Example 4.2. *At the beginning of an experiment, there are* 100 *bacteria. After* 2 *hours, there are* 300 *bacteria. If the bacteria follow and exponential growth pattern, find:*

(a) The number of bacteria $N(t)$ after t hours.

(b) The doubling time for the model.

(c) The differential equation which models this growth pattern.

(d) The amount of time it takes for the number of bacteria to reach 700.

(e) The number of bacteria after 8 *hours.*

Solution. (a) We know $N(t) = N_0 e^{kt}$ (see 1(a) of the summary above). Since there are 100 bacteria at the start, $N_0 = 100$, so all we need is k. To find this, we will use the only other thing we know, which is $N(2) = 300$:

$$\begin{aligned} 300 = 100e^{k\cdot 2} &\implies 3 = e^{2k} \\ &\implies \ln(3) = 2k \quad \text{(take ln of both sides)} \\ &\implies k = \frac{\ln(3)}{2}. \end{aligned}$$

Therefore, $\boxed{N(t) = 100e^{\frac{\ln(3)}{2}t}}$.

(b) The doubling time is $\ln(2)/k$, so it is

$$\frac{\ln(2)}{\ln(3)/2} = \boxed{\frac{2\ln(2)}{\ln(3)} \text{ hours}}.$$

(c) The differential equation is $N'(t) = kN(t)$ since this is exponential growth, so $\boxed{N'(t) = \frac{\ln(3)}{2}N(t)}$.

(d) We want to find when $N(t) = 700$:

$$\begin{aligned} 700 = 100e^{\frac{\ln(3)}{2}t} &\implies 7 = e^{\frac{\ln(3)}{2}t} \\ &\implies \ln(7) = \frac{\ln(3)}{2}t \quad \text{(take ln of both sides)} \\ &\implies \boxed{t = \frac{2\ln(7)}{\ln(3)} \text{ hours}}. \end{aligned}$$

(e) The number of bacteria after 8 hours is just $N(8)$, so $\boxed{100e^{\frac{\ln(3)}{2}\cdot 8}}$.

Example 4.3. *Suppose that the half-life of a certain radioactive substance is* 3 *days. Find:*

(a) The amount $W(t)$ of the substance after t days if the initial amount is W_0.

(b) The differential equation which models this situation.

(c) The amount of time it takes for 70% *of the substance to decay.*

Solution. (a) Since the half-life is $\ln(2)/k$, we know $3 = \frac{\ln(2)}{k}$, or $k = \frac{\ln(2)}{3}$. Hence, $\boxed{W(t) = W_0 e^{-\frac{\ln(2)}{3}t}}$. The negative is because this is exponential decay.

(b) Now, the differential equation is $W'(t) = -kW(t)$, or $\boxed{W'(t) = -\dfrac{\ln(2)}{3}W(t)}$.

(c) If 70% of the substance has decayed, then 30% is left, and 30% of the initial W_0 is $.3W_0$. So we want to solve:

$$\begin{aligned}
.3W_0 = W_0 e^{-\frac{\ln(2)}{3}t} \quad &\Longrightarrow \quad .3 = e^{-\frac{\ln(2)}{3}t} \\
&\Longrightarrow \quad \ln(.3) = \frac{-\ln(2)}{3}t \quad \text{(take ln of both sides)} \\
&\Longrightarrow \quad \boxed{t = -\frac{3\ln(.3)}{\ln(2)} \text{ days}}.
\end{aligned}$$

Example 4.4. *Suppose we have two radioactive substances X and Y. The decay constant for substance X is* 2 *hrs*$^{-1}$ *and the decay constant for substace Y is* 6 *hrs*$^{-1}$*. A certain experiment requires you to have equal amounts of substances X and Y, but you have* 50 *grams of substance X and* 100 *grams of substance Y. How long do you have to wait before you can run the experiment?*

Solution. Let $W_X(t)$ denote the amount of substance X after t hrs and $W_Y(t)$ the amount of substance Y after t hours. Since the "k" for X is 2 and 6 for Y, we have

$$W_X(t) = 50e^{-2t} \qquad W_Y(t) = 100e^{-6t},$$

where the 50 and 100 came from the given initial amounts. We need $W_X(t) = W_Y(t)$, so we solve:

$$\begin{aligned}
50e^{-2t} = 100e^{-6t} \quad &\Longrightarrow \quad 2 = \frac{e^{-2t}}{e^{-6t}} \quad \text{(divide by 50 and } e^{-6t} \text{ on both sides)} \\
&\Longrightarrow \quad 2 = e^{-2t-(-6t)} \quad \text{(when you divide expressions with same base, subtract exponents)} \\
&\Longrightarrow \quad 2 = e^{4t} \\
&\Longrightarrow \quad \ln(2) = 4t \quad \text{(take ln of both sides)} \\
&= \quad \boxed{t = \frac{\ln(2)}{4} \text{ hours}}.
\end{aligned}$$

5 Compund Interest and Present Value (7.5)

Summary 5.1. *(1) Principal amount P_0, interest rate r, compounded M times per year: $P(t) = P_0\left(1 + \frac{r}{M}\right)^{Mt}$.*

(2) Principal amount P_0, interest rate r, compounded continuously:

$$P(t) = \lim_{M\to\infty} P_0\left(1 + \frac{r}{M}\right)^{Mt} = P_0 e^{rt}.$$

(3) *The **present value** of P dollars to be received at time t is Pe^{-rt}. This is the amount of money you would need to invest today to get P dollars at time t (assuming interest rate of r, compounded continuously).*

(4) *The present value for income stream $R(t)$ dollars per year continuous for T years is*

$$\int_0^T R(t)e^{-rt}dt.$$

Remark. Notice that in (2) above, when the principal is compounded continuously, it behaves like the exponential functions you met in 7.4. So, for example, the doubling time is still $\ln(2)/r$, which you should be able to get anyway by setting $P(t) = 2P_0$.

Example 5.2. *Suppose Bill deposits \$100 into an account which pays 6% interest. What's the balance in the account after 5 years if the interest is compounded:*

(a) *Twice a year*

(b) *Continuously*

Solution. (a) By (1) above, we know $P(t) = 100\left(1 + \frac{.06}{2}\right)^{2t}$ since $M = 2$. Therefore, in 5 years, the balance will be

$$P(5) = \boxed{100(1.03)^{10} \text{ dollars}}.$$

(b) If it is compounded continuously, the model is $P(t) = 100e^{.06t}$, so in 5 years, the balance will be

$$P(5) = \boxed{100e^{.06 \cdot 5} \text{ dollars}}.$$

Example 5.3. *What is the present value of \$1000 to be paid 5 years in the future if the interest rate is 8%?*

Solution. By (3), it is Pe^{-rt}, so $\boxed{1000e^{-.08 \cdot 5} \text{ dollars}}$.

Example 5.4. *If a principal amount of 100 dollars is compounded continuously at a given interest rate r, it will be worth 200 dollars in 8 years. Find the interest rate.*

Solution. There are two ways to do this problem:

(1) We know $P(t) = P_0e^{rt}$, that $P_0 = 100$, and that $P(8) = 200$. So

$$200 = 100e^{r \cdot 8} \implies 2 = e^{8r} \implies \ln(2) = 8r \implies r = \frac{\ln(2)}{8}.$$

Hence, the interest rate is

$$\boxed{\left(100 \cdot \frac{\ln(2)}{8}\right)\%}.$$

(2) The second approach would be to realize that 8 is just the doubling time (i.e. the time it takes the intial investment to double in value). Even though it is an idea from 7.4, it still applies to this situation. Hence $r = \frac{\ln(2)}{8}$, and so the interest rate is $\left(100 \cdot \frac{\ln(2)}{8}\right)\%$, which is what we found above.

6 Other Exponential Models (7.6)

Summary 6.1. *(1) The general solution for the differential equation $y' = k(y - b)$ is $y = b + Ce^{kt}$.*

(2) *Newton's Law of Cooling: If k is the cooling constant, T_0 the ambient temperature, $T(t)$ the temperature of the object, then $T'(t) = -k(T(t) - T_0)$.*

(3) *Free-fall with air resistance: If $v(t)$ is the velocity of the object, m the mass, k the air resistance constant, g acceleration due to gravity, then*

$$v'(t) = -\frac{k}{m}\left(v(t) + \frac{mg}{k}\right).$$

(4) *Continuous annuity: $P'(t) = r\left(P(t) - \frac{N}{r}\right)$, where $P(t)$ is the balance in the annuity, r interest rate, N the withdrawal rate.*

Remark. For (2)-(4) above, you can use (1) to get the general solution.

Example 6.2. *A hard-boiled egg at 98° Celsius is put into a sink of 18° water. After 5 minutes, the egg's temperature is 38°.*

(a) *Determine the temperature $T(t)$ of the egg at time t minutes after it is put into the sink.*

(b) *How long will it take the egg's temperature to reach 20°?*

Solution. (a) The differential equation for this model is $T'(t) = -k(T(t) - 18)$, and the general solution is $T(t) = 18 + Ce^{-kt}$. We need both C and k. We know the initial temperature $T(0) = 98$, so

$$T(0) = 18 + Ce^0 = 98 \implies C = 80.$$

To get k, we use the fact that $T(5) = 38$:

$$38 = 18 + 80e^{-k\cdot 5} \implies e^{-5k} = \frac{1}{4} \implies -5k = \ln(1/4) = -\ln(4) \implies k = \frac{\ln(4)}{5}.$$

So putting it all together, we get

$$\boxed{T(t) = 18 + 80e^{-\frac{\ln(4)}{5}t}}.$$

(b) We want to know when $T(t) = 20$, so solve:

$$\begin{aligned} 20 = 18 + 80e^{-\frac{\ln(4)}{5}t} \quad &\implies \quad 2 = 80e^{-\frac{\ln(4)}{5}t} \\ &\implies \quad e^{-\frac{\ln(4)}{5}t} = \frac{1}{40} \\ &\implies \quad -\frac{\ln(4)}{5}t = \ln(1/40) = -\ln(40) \\ &\implies \quad t = \frac{5\ln(40)}{\ln(4)}. \end{aligned}$$

So the answer is $\boxed{\frac{5\ln(40)}{\ln(4)} \text{ minutes}}$.

Example 6.3. *Suppose you have a cup of coffee with cooling constant $k = .09$ min^{-1}. It is placed in a room of temperature 20° C.*

(a) *At what rate is the temperature changing when the temperature of the coffee is 80° C.*

(b) *If the coffee is served at 90° C, how long will it take to reach 30° C.*

(c) *At what time will the temperature reach 15°.*

Solution. (a) The differential equation is $T'(t) = -k(T(t) - T_0) = -.09(T(t) - 20)$. So if $T(t) = 80$, then

$$T'(t) = -.09(80 - 20) = \boxed{-.09 \cdot 60 \text{ °C/min}}.$$

(b) The general solution is $T(t) = 20 + Ce^{-kt} = 20 + Ce^{-.09t}$. First, since $T(0) = 90$, we can solve for C:

$$90 = 20 + Ce^{-.09 \cdot 0} \implies 90 = 20 + C \implies C = 70,$$

so $T(t) = 20 + 70e^{-.09t}$. We want to know when $T(t) = 30$, so

$$30 = 20 + 70e^{-.09t} \implies \frac{1}{7} = e^{-.09t} \implies -.09t = \ln(1/7) = -\ln(7) \implies t = \boxed{\frac{\ln(7)}{.09} \text{ minutes}}.$$

(c) In theory, you would set $T(t) = 15$, so

$$15 = 20 + 70e^{-.09t} \implies -\frac{5}{70} = e^{-.09t}.$$

However, you cannot take logs because you cannot have the log of a negative number. That tells us there is no t which works. We should have seen this immediately because the temperature should never go below the temperature of the ambient space, which in this case is 20.

Example 6.4. *Suppose that a skydiver jumps out of an airplane. If the terminal velocity is -98 m/s, find the velocity of the skydiver after 15 seconds. Assume $k = 8$ kg/s.*

Solution. Here, $v(t) = -\frac{mg}{k} + Ce^{-(k/m)t}$. We don't know m yet. However, terminal velocity is $-mg/k$, so

$$-98 = -\frac{m \cdot 9.8}{8} \implies m = \frac{98 \cdot 8}{9.8} = 80.$$

So the skydiver's mass is 80 kg, and $v(t) = -\frac{80g}{8} + Ce^{-(8/80)t}$. We also need C, but we know $v(0) = 0$, so

$$-\frac{80g}{8} + Ce^0 = 0 \implies C = 10g = 10(9.8) = 98.$$

Hence $v(t) = -10g + 98e^{-t/10} = -98 + 98e^{-t/10}$. After 15 seconds, the velocity is

$$v(15) = \boxed{-98 + 98e^{-15/10} \text{ m/s}}.$$

Example 6.5. *What is the minimum deposit P_0 necessary that will allow an annuity to pay out 1000 dollars/year indefinitely if it earns interest at a rate of 4%.*

Solution. We know $P(t) = \frac{N}{r} + Ce^{rt} = \frac{1000}{.04} + Ce^{.04t}$. The annuity will pay out indefinitely if $C \geq 0$. To see why, notice that if $C < 0$, then $P(t)$ is decreasing, and so will eventually hit 0, which is not what we want. However, if $C \geq 0$, then at worst $P(t)$ is constant, which is ok. If the initial amount is P_0, then

$$P_0 = \frac{1000}{.04} + C \implies C = P_0 - \frac{1000}{.04}.$$

Since we want $C \geq 0$, this means $P_0 \geq \boxed{\frac{1000}{.04} \text{ dollars}}$.

Example 6.6. *A skydiver jumps out of an airplane with zero initial velocity. Before the parachute opens, the skydiver's velocity satisfies $v'(t) = -g - 2v(t)$, and after the parachute opens, the skydiver's velocity satisfies $v'(t) = -g - v$. If the parachute opens 10 seconds after the initial jump, find the velocity 20 seconds after the initial jump.*

Solution. There are two separate velocity functions here, one before the parachute opens and one after, so let's call them v_b and v_a (for "before" and "after"). We know $v_b' = -g - 2v_b' = -\frac{1}{2}\left(v_b + \frac{g}{2}\right)$. This has general solution

$$v_b(t) = -\frac{g}{2} + Ce^{-(1/2)t}.$$

Now, there is zero initial velocity, so $v_b(0) = 0$, or

$$-\frac{g}{2} + C = 0 \implies C = \frac{g}{2}.$$

Therefore

$$v_b(t) = -\frac{g}{2} + \frac{g}{2}e^{-t/2}.$$

We've exhausted this function for now. Next, we know v_a satisfies $v_a' = -g - v_a = -(v_a + g)$, and this has general solution $v_a(t) = -g + ce^{-t}$. However, we cannot say $v_a(0) = 0$. In fact, here, the initial velocity for v_a should be the velocity of skydiver the moment the parachute opened, which is $v_b(10) = -\frac{g}{2} + \frac{g}{2}e^{-5}$ since the parachute opened 10 seconds after the jump. Now we can solve for c, since $v_a(0) = v_b(10)$:

$$-\frac{g}{2} + \frac{g}{2}e^{-5} = -g + c \implies c = \frac{g}{2} + \frac{g}{2}e^{-5}.$$

So

$$v_a(t) = -g + \left(\frac{g}{2} + \frac{g}{2}e^{-5}\right)e^{-t}.$$

Finally, we want the velocity 20 seconds after the initial jump, which is 10 seconds after the parachute opened, so the final answer is

$$v_a(10) = \boxed{-g + \left(\frac{g}{2} + \frac{g}{2}e^{-5}\right)e^{-10} \text{ m/s}}.$$

7 L'Hopital's Rule (7.7)

There are seven indetermine forms, but only three you need to know: $\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty$. If you have a limit that produces one of these forms (and only one of these forms), you will eventually be using L'Hopital's rule. It is extremely important to note that if it is not one of these three, then you will not be using L'Hopital's rule.

Summary 7.1. *(1) Indeterminate Forms:*

(a) If $\lim_{x\to a} \frac{f(x)}{g(x)}$ is either a $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form, and assuming $g'(a) \neq 0$, then we can say

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

which is L'Hopital's rule.

(b) If $\lim_{x\to a} f(x)g(x)$ is a $0 \cdot \infty$ form, then you reduce to case (a) by "flipping" one of the terms. This amounts to writing

$$\lim_{x\to a} f(x)g(x) = \lim_{x\to a} \frac{f(x)}{\frac{1}{g(x)}} \quad \text{or} \quad \lim_{x\to a} \frac{g(x)}{\frac{1}{f(x)}}.$$

(2) Growth of functions: We write $f(x) << g(x)$ if $\lim_{x\to\infty} \frac{g(x)}{f(x)} = \infty$ (equivalently $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$).

Remark. First, in (1) above, a can be taken to be ∞. Second, when flipping one of the terms in the $0 \cdot \infty$ case, we generally avoid flipping logs. On the other hand, flipping exponentials is generally favorable.

Example 7.2. *Calculate $\lim_{x\to 0} \frac{1-e^x}{x-x^2}$.*

Solution. When we plug in 0 into the function, we get $0/0$, so we can use L'Hopital's rule immediately:

$$\lim_{x\to 0} \frac{1-e^x}{x-x^2} \overset{\text{LH}}{=} \lim_{x\to 0} \frac{-e^x}{1-2x} = -\frac{e^0}{1-2(0)} = \boxed{-1},$$

where the LH just means we are applying L'Hopital at that step.

Example 7.3. *Calculate* $\lim_{x\to 1} \frac{\ln(x)}{x^3-1}$

Solution. Again, we plug in our point (in this case $x = 1$) and find a $0/0$ form, so L'Hopital's rule yields:

$$\lim_{x\to 1} \frac{\ln(x)}{x^3-1} \overset{\text{LH}}{=} \lim_{x\to 1} \frac{1/x}{3x^2} = \boxed{\frac{1}{3}}.$$

Example 7.4. *Calculate* $\lim_{x\to\infty} \frac{e^x}{x+\ln(x)}$

Solution. Since both the numerator and denominator approach ∞, this is an ∞/∞ limit. We can apply L'Hopital:

$$\lim_{x\to\infty} \frac{e^x}{x+\ln(x)} \overset{\text{LH}}{=} \lim_{x\to\infty} \frac{e^x}{1+\frac{1}{x}}.$$

Now $e^x \to \infty$ and $1 + \frac{1}{x} \to 1$ as $x \to \infty$, so the limit becomes $\infty/1 = \boxed{\infty}$.

Example 7.5. *Calculate* $\lim_{x\to\infty} \frac{\ln(x^2+1)}{x+\sqrt{x}}$.

Solution. Now, this is an ∞/∞ form since $\ln(x^2 + 1)$ and $x + \sqrt{x}$ both approach ∞ as $x \to \infty$. So we can use L'Hopital:

$$\lim_{x\to\infty} \frac{\ln(x^2+1)}{x+\sqrt{x}} \overset{\text{LH}}{=} \lim_{x\to\infty} \frac{\frac{2x}{x^2+1}}{1+\frac{1}{2\sqrt{x}}}.$$

Now, we want to clean this up a bit, so let's clear denominators:

$$\begin{aligned}
\lim_{x\to\infty} \frac{\frac{2x}{x^2+1}}{1+\frac{1}{2\sqrt{x}}} &= \lim_{x\to\infty} \frac{\frac{2x}{x^2+1}}{1+\frac{1}{2\sqrt{x}}} \cdot \frac{2\sqrt{x}(x^2+1)}{2\sqrt{x}(x^2+1)} \\
&= \lim_{x\to\infty} \frac{4x\sqrt{x}}{2\sqrt{x}(x^2+1)+(x^2+1)} \\
&= \lim_{x\to\infty} \frac{4x^{3/2}}{2x^{5/2}+2\sqrt{x}+x^2+1}.
\end{aligned}$$

Now this last limit is also ∞/∞, so you could do L'Hopital's rule, in theory. But you learned in 31A to do limits like this by looking at dominant terms, which in this case gives

$$\frac{4x^{3/2}}{2x^{5/2}} = \frac{2}{x} \to 0$$

as $x \to \infty$. Therefore the limit is $\boxed{0}$.

Example 7.6. *Evaluate* $\lim_{x\to 0^+} x^2 \ln(x)$.

Solution. Since $\ln(x) \to -\infty$, this is a $0 \cdot \infty$ form. As stated in the remark following the summary above, we generally do not flip the logs, so we will write

$$\lim_{x\to 0^+} x^2 \ln(x) = \lim_{x\to 0} \frac{\ln(x)}{\frac{1}{x^2}}.$$

Now this is ∞/∞:

$$\lim_{x\to 0} \frac{\ln(x)}{\frac{1}{x^2}} \overset{\text{LH}}{=} \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{2}{x^3}} = \lim_{x\to 0} \frac{1}{x} \cdot -\frac{x^3}{2} = \lim_{x\to 0} -\frac{x^2}{2} = \boxed{0}.$$

Example 7.7. *Show* $\lim_{x\to\infty} x^2 e^{-x} = 0$ *and conclude* $x^2 << e^x$.

Solution. Since $e^{-x} \to 0$ as $x \to \infty$, this is technically $0 \cdot \infty$. We usually flip exponentials, but thankfully it is a little easier here since $e^{-x} = \frac{1}{e^x}$, so the limit is

$$\lim_{x\to\infty} \frac{x^2}{e^x} \overset{\text{LH}}{=} \lim_{x\to\infty} \frac{2x}{e^x} \overset{\text{LH}}{=} \lim_{x\to\infty} \frac{2}{e^x} = \boxed{0}.$$

The use of L'Hopital's rule in both cases is justified by the fact that the new limits were ∞/∞. Since $\lim_{x\to\infty} \frac{x^2}{e^x} = 0$, by definition, $x^2 << e^x$.

8 Integration by Parts (8.1)

You use integration by parts to undo the product rule. The formula is

$$\boxed{\int u dv = uv - \int v du}.$$

The problem will always be determining what to make u and what to make dv. The phrase you can remember is LIATE:

Logs **I**nverse trig **A**lgebraic **T**rig **E**xponentials.

This gives you a priority listing: logs get highest priority for the choice of u, then inverse trig functions, then algebraic expressions (e.g. x, x^2, polynomials, etc.), then trig functions, and lastly exponentials. Let's look at this in practice:

Example 8.1. *Evaluate* $\int x\cos(x)dx$.

Solution. So first, you recognize this as a by-parts problem because you don't know the antiderivative immediately and there is no clear u-substitution to make. The product of functions is also a clue. So we have to decide on u and dv: x is an algebraic expression and $\cos(x)$ is trig, so x gets priority for u. So let $u = x$, $dv = \cos(x)dx$. Then $du = dx$, and $v = \sin(x)$ (take the antiderivative of dv to get v):

$$u = x \quad dv = \cos(x)dx,$$

$$du = dx \quad v = \sin(x).$$

Therefore, the formula says the integral is

$$uv - \int v du = \overbrace{x}^{u}\overbrace{\sin(x)}^{v} - \int \overbrace{\sin(x)}^{v}\overbrace{dx}^{du} = \boxed{x\sin(x) + \cos(x) + C}.$$

Example 8.2. *Evaluate* $\int xe^{2x}dx$.

Solution. Again, the product of functions is a clue for by parts, as is the lack of a clear substitution choice. This time, we have x, an algebraic expression, and e^{2x}, an exponential, so x gets priority for u:

$$u = x \quad dv = e^{2x}dx,$$

$$du = dx \quad v = \frac{1}{2}e^{2x}.$$

So

$$\int xe^{2x}dx = \overbrace{x}^{u} \cdot \overbrace{\frac{1}{2}e^{2x}}^{v} - \int \overbrace{\frac{1}{2}e^{2x}}^{v}\overbrace{dx}^{du}.$$

Since the last integral is $\frac{1}{4}e^{2x}$, we get an answer of

$$\int xe^{2x}dx = \boxed{\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C}.$$

Example 8.3. *Evaluate* $\int x^2e^{2x}dx$

Solution. Here, x^2 is algebraic, e^{2x} is an exponential, so we will let $u = x^2$:

$$u = x^2 \quad dv = e^{2x}dx,$$

$$du = 2xdx \quad v = \frac{1}{2}e^{2x}.$$

Therefore:
$$\int x^2e^{2x}dx = \frac{1}{2}x^2e^{2x} - \int xe^{2x}dx,$$
since the 2 and 1/2 cancel in the vdu integral. To do this new integral, we need by parts *again.* Thankfully we just did this integral in the previous example, so using that answer we get
$$\boxed{\frac{1}{2}x^2e^{2x} - \left(\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right) + C}.$$

Remark. The point of this example was to point out we can do by parts as many times as we need to.

Example 8.4. *Evaluate* $\int_1^2 x^3\ln(x)dx$.

Solution. Having bounds doesn't change anything. Here, x^3 is algebraic, $\ln(x)$ is a logarithm, so $\ln(x)$ is our choice of u:
$$u = \ln(x) \quad dv = x^3dx,$$
$$du = \frac{1}{x}dx \quad v = \frac{1}{4}x^4.$$
Hence,
$$\int_1^2 x^3\ln(x)dx = \frac{1}{4}x^4\ln(x)\Big|_1^2 - \int_1^2 \frac{1}{4}x^4\cdot\frac{1}{x}dx = \frac{1}{4}x^4\ln(x)\Big|_1^2 - \int_1^2\frac{1}{4}x^3dx = \frac{1}{4}x^4\ln(x) - \frac{1}{16}x^4\Big|_1^2.$$
Now evaluating the bounds, we get
$$\left(\frac{1}{4}\cdot 2^4\ln(2) - \frac{1}{16}\cdot 2^4\right) - \left(\frac{1}{4}\cdot 1^4\ln(1) - \frac{1}{16}\cdot 1^4\right) = \boxed{4\ln(2) - \frac{15}{16}},$$
since $\ln(1) = 0$.

Example 8.5. *Evaluate* $\int \tan^{-1}(x)dx$.

Solution. This is one of the tricks of by-parts integration: knowing to do it even with only one term in the integrand. The trick is to write the integral as $\int \tan^{-1}(x)\cdot 1dx$, so now we do have two terms. Since $\tan^{-1}(x)$ is inverse trig and 1 is an algebraic expression, we will let $u = \tan^{-1}(x)$:
$$u = \tan^{-1}(x) \quad dv = 1dx,$$
$$du = \frac{1}{1+x^2}dx \quad v = x,$$
and so the integral becomes
$$x\tan^{-1}(x) - \int\frac{x}{1+x^2}dx.$$
Now, to evaluate the new integral, we can do a u-sub, so (ignoring the previous u) we will let $u = 1+x^2$, $du = 2xdx$, so $xdx = \frac{1}{2}du$. This means
$$\int\frac{x}{1+x^2}dx = \frac{1}{2}\int\frac{du}{u} = \frac{1}{2}\ln|u| = \frac{1}{2}\ln|1+x^2|.$$
Strictly speaking, we don't need absolute values, but we can leave them in. Putting all of this together, the answer becomes
$$\boxed{x\tan^{-1}(x) + \frac{1}{2}\ln|1+x^2| + C}.$$

Example 8.6. *Evaluate* $\int x^3\sin(x^2)dx$.

Solution. Your first thought for this integral would probably be to do by parts. However, if we were following LIATE, we would need to let $u = x^3$ (the algebraic expression), and so $dv = \sin(x^2)dx$. The problem is we can't find v since we don't know the antiderivative of $\sin(x^2)$. That tells us that by parts is probably not the best way to go (at least at first).

Once you determine this, your next thought might be u-sub. Indeed, this is one of those tricky u-subs. We will let $u = x^2$, and $du = 2xdx$, so $\frac{1}{2}du = xdx$. The problem is we have x^3, not just x. So write

$$\int x^3 \sin(x^2)dx = \int x \cdot x^2 \sin(x^2)dx.$$

Now the xdx becomes $\frac{1}{2}du$, and the $x^2\sin(x^2)$ becomes $u\sin(u)$. So we have transformed the integral as

$$\int x^3 \sin(x^2)dx = \frac{1}{2}\int u\sin(u)du.$$

For this new integral, we need by parts. Since u is algebraic and $\sin(u)$ is trig, we will let $U = u$ (the capital U being because we have already used lower case u):

$$U = u \quad dv = \sin(u)du,$$

$$dU = du \quad v = -\cos(u).$$

So we have

$$\frac{1}{2}\int u\sin(u)du = \frac{1}{2}\left[-u\cos(u) - \int -\cos(u)du\right] = \frac{1}{2}\left[-u\cos(u) + \sin(u)\right] + C.$$

Lastly, substitute in $u = x^2$:

$$\boxed{\frac{1}{2}\left[-x^2\cos(x^2) + \sin(x^2)\right] + C}.$$

Example 8.7. *Evaluate* $\int e^x \cos(x)dx$.

Solution. This is the trickiest one of all. We have an exponential and a trig function, so just to stay consistent with LIATE we will let $u = \cos(x)$ (though, in reality, it doesn't matter here):

$$u = \cos(x) \quad dv = e^x dx,$$

$$du = -\sin(x)dx \quad v = e^x.$$

So we have

$$\int e^x \cos(x)dx = e^x\cos(x) - \int -\sin(x)e^x dx = e^x\cos(x) + \int e^x\sin(x)dx.$$

For this new integral, we need by parts again. We will let $u = \sin(x)$ this time (note: if you make $u = e^x$ this time, you will undo the work that you did to get to this point):

$$u = \sin(x) \quad dv = e^x dx,$$

$$du = \cos(x)dx \quad v = e^x.$$

So

$$\int e^x\sin(x)dx = e^x\sin(x) - \int e^x\cos(x)dx.$$

You should notice that this is the same integral we started with, so it may seem like we went in a circle, but we didn't. What we have is

$$\int e^x\cos(x)dx = e^x\cos(x) + e^x\sin(x) - \int e^x\cos(x)dx. \tag{1}$$

The integral is what we're solving for, so treat it like a variable. If we move the integral on the right to the other side (by adding it to both sides), we get

$$2\int e^x \cos(x)dx = e^x \cos(x) + e^x \sin(x),$$

so dividing by 2 gives:

$$\int e^x \cos(x)dx = \boxed{\frac{e^x \cos(x) + e^x \sin(x)}{2}}.$$

Remark. If this last step bothers you, you could say: Let $I = \int e^x \cos(x)dx$. We will try and solve for I. Equation (1) becomes:

$$I = e^x \cos(x) + e^x \sin(x) - I \implies 2I = e^x \cos(x) + e^x \sin(x) \implies I = \frac{e^x \cos(x) + e^x \sin(x)}{2},$$

which is the same as the answer we got.

9 Trig Integals (8.2)

The only thing you need to know from this section is how to integrate things like

$$\int \sin^n(x)\cos^m(x)dx,$$

where either m or n is odd. Just always remember the identity

$$\boxed{\sin^2(x) + \cos^2(x) = 1}.$$

Example 9.1. *Integrate* $\sin^3(x)\cos(x)$.

Solution. In the case one of the powers happens to be 1, no extra work is necessary. Just do a u-sub: $u = \sin(x)$, $du = \cos(x)dx$, so the integral

$$\int \sin^3(x)\cos(x)dx = \int u^3 du = \frac{u^4}{4} + C = \boxed{\frac{\sin^4(x)}{4} + C}.$$

Example 9.2. *Integrate* $\sin^3(x)\cos^2(x)$.

Solution. Now we can't do a simply u-sub, since we have a $\cos^2(x)$ and not just $\cos(x)$. The trick is to break up the $\sin^3(x)$ (or, more generally, the trig function with an odd power):

$$\int \sin^3(x)\cos^2(x)dx = \int \sin(x)\cdot\sin^2(x)\cdot\cos^2(x)dx = \int \sin(x)(1-\cos^2(x))\cos^2(x)dx,$$

where for this last step we used the pythagorean identity above. Now we can do a u-sub with $u = \cos(x)$, $du = -\sin(x)dx$, so $-du = \sin(x)dx$. Thus, the integral becomes

$$-\int (1-u^2)u^2 du = -\int u^2 - u^4 du = -\frac{u^3}{3} + \frac{u^5}{5} + C = \boxed{-\frac{\cos^3(x)}{3} + \frac{\cos^5(x)}{5} + C}.$$

Example 9.3. *Integrate* $\sin^5(x)\cos^6(x)$.

Solution. Again, we break down the function with the odd power: $\sin^5(x) = \sin(x)\cdot\sin^4(x) = \sin(x)\cdot(\sin^2(x))^2 = \sin(x)\cdot(1-\cos^2(x))^2$. Therefore,

$$\int \sin^5(x)\cos^6(x)dx = \int \sin(x)(1-\cos^2(x))^2\cdot\cos^6(x)dx.$$

Now do $u = \cos(x)$, $du = -\sin(x)dx$, and again $-du = \sin(x)dx$. This yields

$$-\int (1-u^2)^2 u^6 du.$$

The least painful way to do this is just distribute:

$$-\int (1-u^2)^2 u^6 du = -\int (1-2u^2+u^4)u^6 du = -\int u^6 - 2u^8 + u^{10} du = -\frac{u^7}{7} + \frac{2u^9}{9} - \frac{u^{11}}{11} + C.$$

Plugging in $u = \cos(x)$ back in gives

$$\boxed{-\frac{\cos^7(x)}{7} + \frac{2\cos^9(x)}{9} - \frac{\cos^{11}(x)}{11} + C}.$$

Example 9.4. *Integrate* $\sin^3(x)\cos^5(x)$.

Solution. In this case, both powers are odd, so it doesn't much matter which one you break down. In general, it is probably easier manipulate the lower odd degree: $\sin^3(x) = \sin(x)\cdot\sin^2(x) = \sin(x)(1-\cos^2(x))$. So we have

$$\int \sin(x)(1-\cos^2(x))\cos^5(x)dx.$$

After the u-sub we have done the previous two times with $u = \cos(x)$, we get

$$-\int (1-u^2)u^5 du = -\int u^5 - u^7 du = -\frac{u^6}{6} + \frac{u^8}{8} + C = \boxed{-\frac{\cos^6(x)}{6} + \frac{\cos^8(x)}{8} + C}.$$

10 Trigonometric Substitution (8.3)

Trig sub helps tackle integrals involving $ax^2 - b$, $ax^2 + b$, and $a - bx^2$. We just need to remember the two identities:

$$\boxed{\sin^2(x) + \cos^2(x) = 1 \quad \tan^2(x) + 1 = \sec^2(x)}.$$

Example 10.1. *Integrate* $\sqrt{1-x^2}$

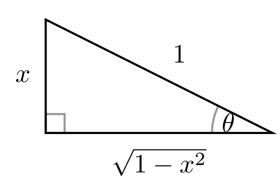
Solution. We know that $1 - \sin^2(x) = \cos^2(x)$, so we will let $x = \sin(\theta)$. Then $dx = \cos(\theta)d\theta$, and so

$$\int \sqrt{1-x^2}dx = \int \sqrt{1-\sin^2(\theta)}\cos(\theta)d\theta = \int \cos^2(\theta)d\theta.$$

To integrate $\cos^2(\theta)$, we use the identity $\cos(2\theta) = 2\cos^2(\theta) - 1$, so $\cos^2(\theta) = \frac{1}{2}(\cos(2\theta) - 1)$. Hence

$$\int \cos^2(\theta)d\theta = \frac{1}{2}\int \cos(2\theta) - 1 d\theta = \frac{1}{4}\sin(2\theta) - \frac{1}{2}\theta + C.$$

Now, we want to go back to x's. We know $\theta = \sin^{-1}(x)$ by our original subsitution. Recall that $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$. We know $\sin(\theta) = x = \frac{x}{1}$, and we can get $\cos(\theta)$ from the following triangle:



To get the sides of the triangle, we used the fact that $\sin(\theta)$ is opposite/hypotenuse, and since $\sin(\theta) = x/1$, we can call x the opposite side and 1 the hypotenuse. We then get the adjacent side using the Pythagorean theorem. So $\sin(2\theta) = 2x\sqrt{1-x^2}$, and our answer becomes

$$\boxed{\frac{1}{4}\cdot 2x\sqrt{1-x^2} - \frac{1}{2}\sin^{-1}(x) + C}.$$

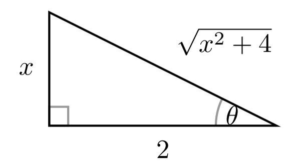
Remark. We saw above that for integrals involving $1-x^2$, we use $x=\sin(\theta)$. For those involving x^2-1, we will use $x=\sec(\theta)$, and for those involving x^2+1, we will use $x=\tan(\theta)$.

Example 10.2. *Evaluate* $\int \frac{dx}{(x^2+4)^{3/2}}$.

Solution. We want to use $\tan^2(x)+1=\sec^2(x)$, but we have x^2+4, not x^2+1. But notice that if $x=2\tan(\theta)$, then $x^2+4=4\tan^2(\theta)+4=4\sec^2(\theta)$. So let $x=2\tan(\theta)$, so $dx=2\sec^2(\theta)d\theta$. Then

$$\int \frac{dx}{(x^2+4)^{3/2}} = \int \frac{2\sec^2(\theta)d\theta}{(4\sec^2(\theta))^{3/2}} = \int \frac{2\sec^2(\theta)d\theta}{(8\sec^3(\theta))} = \frac{1}{4}\int \frac{1}{\sec(\theta)}d\theta = \frac{1}{4}\int \cos(\theta)d\theta.$$

Integrating, we get $\frac{1}{4}\sin(\theta)+C$. Now draw the triangle. Since $x=2\tan(\theta)$, $\tan(\theta)=x/2$, and this is opposite/adjacent, so we can call the opposite side x and the adjacent side 2. The hypotenuse is then $\sqrt{x^2+4}$ by the Pythagorean theorem:



We see that if $x=2\tan(\theta)$, then $\sin(\theta)=\frac{x}{\sqrt{x^2+4}}$ (opposite/hypotenuse), so the answer is

$$\frac{1}{4}\frac{x}{\sqrt{x^2+4}}+C.$$

Remark. Here we needed to adjust for the 4 in our substitution. There are formulas for this, but just think about the identities you are using and what constants you need to get them to work.

Example 10.3. *Integrate* $\frac{1}{\sqrt{5x^2-4}}$.

Solution. Now we see the ax^2-b form, so we will let x be something times $\sec(\theta)$. To make it easier to work with, factor out a 5 from the denominator:

$$\int \frac{dx}{\sqrt{5\left(x^2-\frac{4}{5}\right)}} = \frac{1}{\sqrt{5}}\int \frac{dx}{\sqrt{x^2-\frac{4}{5}}}.$$

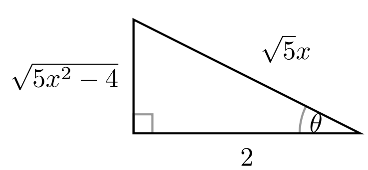
Now, we will let $x=\frac{2}{\sqrt{5}}\sec(\theta)$, so that

$$x^2-\frac{4}{5}=\frac{4}{5}\sec^2(\theta)-\frac{4}{5}=\frac{4}{5}(\sec^2(\theta)-1)=\frac{4}{5}\tan^2(\theta).$$

Don't forget $dx=\frac{2}{\sqrt{5}}\sec(\theta)\tan(\theta)d\theta$. Thus, the integral becomes

$$\frac{1}{\sqrt{5}}\int \frac{\frac{2}{\sqrt{5}}\sec(\theta)\tan(\theta)d\theta}{\frac{2}{\sqrt{5}}\tan(\theta)} = \frac{1}{\sqrt{5}}\int \sec(\theta)d\theta = \frac{1}{\sqrt{5}}\ln|\sec(\theta)+\tan(\theta)|+C.$$

Finally, we want this back in terms of x. Write $\sec(\theta)=\frac{\sqrt{5}x}{2}$, so the triangle looks like:



Here we used that $\sec(\theta)$ is hypotenuse/adjacent, and got the remaining side by Pythagorean theorem. So $\sec(\theta)=\frac{\sqrt{5}}{2}x$ (which we knew already), and $\tan(\theta)=\frac{\sqrt{5x^2-4}}{2}$. Thus, our answer is

$$\boxed{\frac{1}{\sqrt{5}}\ln\left|\frac{\sqrt{5}}{2}x+\frac{1}{2}\sqrt{5x^2-4}\right|+C}.$$

Remark. The trickiest part of trig sub is knowing what to make x, so just make sure you understand what identities you are using for each of the three types demonstrated above.

11 Partial Fractions (8.5)

We begin with words of caution:

CAUTION. The first, and most important, thing to note about partial fractions is that you only use it when you can factor the denominator. For example, the integral

$$\int \frac{dx}{x^2 - 2x + 6}$$

is not a partial fractions problem because the denominator cannot be factored (such things are called *irreducible*).

CAUTION. The second important point is that even if the denominator is factorable, you need the degree of the numerator to be strictly less than the degree of the denominator. If it is not, you must do long division first.

Now that we have made note of these two facts, we can actually describe the method. The idea is that we would like to break down fractions such as $\frac{1}{x^2-x}$ into easier pieces:

$$\frac{1}{x(x-1)} = \frac{-1}{x} + \frac{1}{x-1}.$$

We will do this by making a guess, i.e. assuming we can write

$$\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1},$$

and then solving for A and B. However, first you need to learn what to guess. There are several cases, all having to do with how the denominator factors. For example, for an integral like

$$\int \frac{dx}{(x-1)^2(x^2+1)^2},$$

the key numbers you want to pay attention to are: the degree of each irreducible term (in this case, the $x-1$ and the x^2+1), and the powers they are raised two (in this case, both 2's). You will then decompose them factor by factor. Let's look at the cases:

(1) **Linear factor to the first power**: If you had something like $x - 1$ in the denominator, then all you need in the decomposition is $\frac{\text{constant}}{x-1}$. So, for example, we wanted to break down $\frac{1}{(x-2)(x-3)}$, our guess would be $\frac{A}{x-2} + \frac{B}{x-3}$, since both $x - 2$ and $x - 3$ are linear, and each is raised only to the first power.

(2) **Linear factors to a higher power**: If you have something like $(x + 1)^2$ in the denominator, then the fact that $x + 1$ is still only linear tells us we only need constants in the numerator. However, we must account for all powers of $(x+1)$, so we would need to have $\frac{A}{x+1} + \frac{B}{(x+1)^2}$. An example: if we had $\frac{1}{(x-2)^2(x-3)}$, we would notice we still have linear terms, but now the $x - 2$ is squared. We deal with this by just remembering to account for every power of our linear terms. So we would have to break it up as

$$\frac{1}{(x-2)^2(x-3)} = \frac{?}{x-2} + \frac{?}{(x-2)^2} + \frac{?}{x-3}.$$

Since each term was still linear, we only put constants on top, so

$$\frac{1}{(x-2)^2(x-3)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x-3}.$$

(3) **Quadratic Factor to the first power**: If the denominator contains an irreducible quadratic to the first power, such as x^2+9, then you only need one term, but instead of a constant on top, you put a degree 1 term, i.e. $\frac{Ax+B}{x^2+9}$. Consider $\frac{x}{(x-1)^3(x^2+1)}$. We can deal with the $(x-3)^2$ as in (2). However, since the x^2+1 is an irreducible quadratic (i.e. it cannot be factored), we need to account for this by putting a linear term on top. The terms on top of the $x-3$ fractions will all be constants since $x-3$ is degree 1 (so basically, it is one less degree on top).

$$\frac{x}{(x-1)^3(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{Dx+E}{x^2+1}.$$

They point of this case is to notice the $Dx+E$ on top of the x^2+1.

(4) **Quadratic Factors to higher powers**: We do the same thing as in (2), but remember you need degree 1 terms on top. So for $\frac{1}{(x-1)^2(x^2+1)^2}$, we have linear power squared, so that is dealt with in case (2). For the $(x^2+1)^2$, we will need to account for both (x^2+1) and $(x^2+1)^2$ as in (2), but put linear terms on top:

$$\frac{1}{(x-1)^2(x^2+1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2}.$$

Doing some examples will probably help, since we haven't actually worked anything out.

Example 11.1. *Find the partial fraction decomposition for* $\frac{1}{(x-1)(x+1)}$.

Solution. We must focus on the factors in the denominator:

(a) The $x-1$: This is a linear term to the first power, so we use (1) above, and the only contribution to the decomposition is $\frac{A}{x-1}$.

(b) The $x+1$: Still only linear and to the first power, so again use (1). Here, the only term we need is $\frac{B}{x+1}$.

So together we have

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}.$$

Now to solve for A and B, combine the terms on the right side by getting a LCD (it will always be the original denominator):

$$\frac{1}{(x-1)(x+1)} = \frac{A(x+1)}{(x-1)(x+1)} + \frac{B(x-1)}{(x-1)(x+1)} = \frac{A(x+1)+B(x-1)}{(x-1)(x+1)}.$$

This means $1 = A(x+1) + B(x-1)$. Now there are two approaches to solve for A and B:

(1) Expand the right side and compare coefficients: The right side is $Ax+A+Bx-B$. Combining powers of x gives $(A+B)x+(A-B)$. We need to set this to 1. But if this is going to be true for all x, then the coefficients must be the same. For example, if $ax^2+bx+c = 2x+3$, then $a=0$, $b=2$, and $c=3$. So here, $A+B=0$ since there are no x on the left side, and $A-B=1$, since the constant term is 1. Now you have simultaneous equations:

$$A+B=0,$$
$$A-B=1.$$

If you add them, you get $2A=1$, so $A=1/2$. This means $B=-1/2$. Thus,

$$\boxed{\frac{1}{(x-1)(x+1)} = \frac{1/2}{x-1} + \frac{-1/2}{x+1}}.$$

(2) The other method would be to get to $1 = A(x+1) + B(x-1)$, and say that since this should be true for all x, if we plug in certain values of x the equation is still true. In theory you could plug in *any* value, but some are convenient to choose. For example, if we pick $x = -1$ and plug it in, the $x+1$ term is 0, so $1 = B(-1-1) \implies B = -1/2$, which is exactly what we found above. If we plug in $x = 1$, then the $x-1$ term vanishes, and we get $1 = A(1+1) \implies A = 1/2$.

Example 11.2. *Integrate* $\frac{3x+5}{x^2+3x+2}$.

Solution. Notice that the denominator factors as $\frac{3x+5}{(x+2)(x+1)}$. So this is a partial fractions problem. First, we decompose the fraction. Since all our factors in the denominator are linear and to the first power, it is the easy case:

$$\frac{3x+5}{(x+2)(x+1)} = \frac{A}{x+2} + \frac{B}{x+1} \implies \frac{3x+5}{(x+1)(x+2)} = \frac{A(x+1)+B(x+2)}{(x+1)(x+2)}.$$

Here we skipped a step, but we combined the fractions. So $3x+5 = A(x+1)+B(x+2)$. Plugging in -1 on both sides gives $2 = B(-1+2) \implies B = 2$. Plugging in -2 on both sides yields $-1 = A(-2+1) \implies A = 1$. Therefore

$$\frac{3x+5}{x^2+3x+2} = \frac{1}{x+2} + \frac{2}{x+1}.$$

So

$$\int \frac{3x+5}{x^2+3x+2}dx = \int \frac{1}{x+2} + \frac{2}{x+1}dx = \boxed{\ln|x+2| + 2\ln|x+1| + C}.$$

Example 11.3. *Integrate* $\frac{4x^2+3x+2}{x^2(x+2)}$.

Solution. Now we have a linear factor squared in the x^2 and a linear factor to the first power with $(x+2)$. (Notice we treat x^2 as linear squared instead of an irreducible quadratic, but technically it won't matter.) So the decomposition looks like

$$\frac{4x^2+3x+2}{x^2(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}.$$

Either combine the fractions (with LCD $x^2(x+2)$), or just multiply both sides by $x^2(x+2)$, and we get

$$4x^2+3x+2 = Ax(x+2) + B(x+2) + Cx^2.$$

Plugging in $x = 0$ on both sides gives $2 = 2B \implies B = 1$. Plugging in $x = -2$ on both sides gives $12 = 4C \implies C = 3$. We have run out of convenient values though, so plug in any x to get the final one. Plugging in $x = 1$ gives $9 = A(1)(3) + B(3) + C(1)^2$. But we know B and C, so using these, we get

$$9 = 3A + 3(1) + 3(1)^2 \implies 3 = 3A \implies A = 1.$$

Therefore

$$\int \frac{4x^2+3x+2}{x^2(x+2)}dx = \int \frac{1}{x} + \frac{1}{x^2} + \frac{3}{x+2}dx = \boxed{\ln|x| - \frac{1}{x} + 3\ln|x+2| + C}.$$

CAUTION. Don't just assume every term gives a logarithm. You'll notice that the $1/x^2$ gave a $-1/x$, not an $\ln(x^2)$.

Example 11.4. *Integrate* $\frac{2x^2+x+1}{x(x^2+1)}$.

Solution. Now we have a linear term, x, and an irreducible quadratic in x^2+1. The quadratic means we need a linear term on top:

$$\frac{2x^2+x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} \implies 2x^2+x+1 = A(x^2+1) + (Bx+C)x.$$

Plugging in $x = 0$ on both sides gives $1 = A(1) \implies A = 1$. However, instead of plugging in more values, now might be a good time to try the other method, namely comparing coefficients:

$$2x^2+x+1 = Ax^2 + A + Bx^2 + Cx = (A+B)x^2 + Cx + A.$$

We already know $A = 1$. Comparing the x terms gives $C = 1$. Comparing x^2 terms gives $2 = A + B$, so $B = 1$. Thus,

$$\int \frac{2x^2 + x + 1}{x(x^2+1)} dx = \int \frac{1}{x} + \frac{x+1}{x^2+1} dx.$$

The first term gives $\ln|x|$. To integrate the second term, break it up:

$$\int \frac{x+1}{x^2+1} dx = \int \frac{x}{x^2+1} dx + \int \frac{1}{x^2+1} dx.$$

The second integral is $\tan^{-1}(x)$, and the first requires a u-sub: $u = x^2 + 1$, $du = 2xdx$, so $\frac{1}{2}du = xdx$:

$$\int \frac{x}{x^2+1} dx = \int \frac{1}{2} \frac{du}{u} = \frac{1}{2} \ln|u| = \frac{1}{2} \ln|x^2+1|.$$

Putting it all together, we get

$$\int \frac{2x^2+x+1}{x(x^2+1)} dx = \boxed{\ln|x| + \frac{1}{2} \ln|x^2+1| + \tan^{-1}(x) + C}.$$

Example 11.5. *Integrate* $\frac{x^4+3x^2+1}{x(x^2+1)^2}$.

Solution. This is the last case, since we have an irreducible quadratic, but squared. So the decomposition looks like

$$\frac{x^4+3x^2+1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}.$$

Observe that we had linear terms on top of the x^2+1 terms because x^2+1 is quadratic. This will be a little messy, but combining fractions eventually produces the equation

$$x^4 + 3x^2 + 1 = A(x^2+1)^2 + (Bx+C)(x^2+1) + (Dx+E)x.$$

Here, it might actually be better to compare coefficients. Distributing the entire right side gives:

$$x^4+3x^2+1 = A(x^4+2x^2+1)+Bx^3+Bx+Cx^2+C+Dx^2+Ex = Ax^4+Bx^3+(2A+C+D)x^2+(B+E)x+(A+C).$$

Comparing x^4 coefficients gives $A = 1$. Since there is no x^3 term on the left, $B = 0$. Since there are no x's either, $B + E = 0$, so $E = 0$. Comparing constant terms gives $A + C = 1 \implies C = 0$ since $A = 1$. Finally, since $2A + C + D = 3$, we get $D = 1$ since $A = 1$ and $C = 0$. Thus,

$$\int \frac{x^4+3x^2+1}{x(x^2+1)^2} dx = \int \frac{1}{x} + \frac{x}{(x^2+1)^2} dx = \int \frac{1}{x} dx + \int \frac{x}{(x^2+1)^2} dx.$$

The first integral is $\ln|x|$. The second requires a u-sub with $u = x^2+1$, $du = 2xdx$, meaning $\frac{1}{2}du = xdx$. Therefore,

$$\int \frac{xdx}{(x^2+1)^2} = \int \frac{1}{2} \frac{du}{u^2} = -\frac{1}{2} \cdot \frac{1}{u} = -\frac{1}{2(x^2+1)}.$$

Finally,

$$\int \frac{x^4+3x^2+1}{x(x^2+1)^2} = \boxed{\ln|x| - \frac{1}{2(x^2+1)} + C}.$$

Example 11.6. *Integrate* $\frac{x^2}{x^2-1}$.

Solution. We can factor the denominator as $(x+1)(x-1)$, but the degree on top is the same as the degree in the denominator. As mentioned in the second caution remark in the beginning of the subsection, we must do long division first. Doing long division gives $\frac{x^2}{x^2-1} = 1 + \frac{1}{x^2-1} = 1 + \frac{1}{(x-1)(x+1)}$. [If you have trouble with long division, either look in the textbook or come to office hours and I'd be happy to explain it to you.] So we have

$$\int \frac{x^2}{x^2-1} dx = \int 1 + \frac{1}{x^2-1} dx = x + \frac{1}{x^2-1} dx.$$

Thankfully, in example 2.16, we already worked out the partial fraction decomposition for $\frac{1}{x^2-1}$. So using that, we get

$$x + \int \frac{1}{x^2-1} dx = x + \int \frac{1/2}{x-1} + \frac{-1/2}{x+1} dx = \boxed{x + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C}.$$

12 Improper Integrals and Comparison Test (8.6)

12.1 Improper Integrals

An integral can be improper for two reasons: (1) You have infinite bounds or (2) a discontinuity in the interval (or both).

Example 12.1. *The integral*

$$\int_1^\infty \frac{1}{x} dx$$

is improper because we have an infinite bound. Note that the discontinuity of our function $f(x) = \frac{1}{x}$ *at* $x = 0$ *does not matter since it is not in the interval* $[1, \infty)$*.*

Example 12.2. *The integral*

$$\int_{-\infty}^1 \frac{1}{x+5} dx$$

is improper because we have an infinite bound and *the function* $f(x) = \frac{1}{x+5}$ *is not continuous at* $x = -5$*, which is in our interval.*

When faced with an improper integral, the first thing you need to do is identify all the improper parts. We consider the cases:

(1) **One improper bound**: Suppose you have an integral which is only improper for one reason, and that reason is at the bound. For example, the first example (2.22) above is fine because the only improper part of the integral is the infinite bound, and it is a bound. Example 2.23 is not good because not only do we have the infinite bound, but also the discontinuity inside the interval. The integral

$$\int_{-1}^1 \frac{1}{x} dx$$

is also bad because we have only the discontinuity at $x = 0$, but it happens *inside* the interval $[-1, 1]$. But in the case where we have only one improper bound and we are ok inside the interval, we deal with the improper bound by replacing it by a variable, say M, and taking a limit to the improper bound.

Example 12.3. *Evaluate* $\int_1^\infty \frac{1}{x} dx$

Solution. The ∞ is the problem, so you replace it by some number M, and then we will let M go to infinity:

$$\int_1^\infty \frac{1}{x} dx = \lim_{M\to\infty} \int_1^M \frac{1}{x} dx.$$

You then compute the integral as you normally would:

$$\int_1^M \frac{1}{x} dx = \ln|x| \Big|_1^M = \ln|M| - \ln|1| = \ln|M|.$$

Now take a limit as $M \to \infty$:

$$\lim_{M\to\infty} \int_1^M \frac{1}{x} dx = \lim_{M\to\infty} \ln|M| = \infty.$$

Since we got ∞ and not a number, the integral is said to *diverge*. If we had gotten a number, then the integral is called *convergent*.

Some other worked examples:

Example 12.4. *Evaluate*

$$\int_0^\infty \frac{1}{1+x^2} dx.$$

Solution. This integral is improper because of the infinite bound, and there are no discontinuities. So replace ∞ by M and then take a limit:

$$\begin{aligned}
\int_0^\infty \frac{1}{1+x^2} dx &= \lim_{M\to\infty} \int_0^M \frac{dx}{1+x^2} \\
&= \lim_{M\to\infty} \tan^{-1}(x) \Big|_0^M \\
&= \lim_{M\to\infty} \tan^{-1}(M) - \tan^{-1}(0) \\
&= \lim_{M\to\infty} \tan^{-1}(M) \\
&= \boxed{\frac{\pi}{2}},
\end{aligned}$$

so the integral converges.

Example 12.5. *Evaluate*

$$\int_0^1 \frac{1}{\sqrt{x}} dx.$$

Solution. The integral is bad because the function is not continuous at $x = 0$, but this is the only problem. So put in a number M, and take a limit:

$$\int_0^1 \frac{1}{\sqrt{x}} dx = \lim_{M\to 0^+} \int_M^1 \frac{1}{\sqrt{x}} dx.$$

[You'll notice the limit is from the right: M is supposed to be a number between 0 and 1, and then we are sliding M *down* to 0, so we are going from the right. If it had been in the upper bound, we would go from the left.]

$$\begin{aligned}
\lim_{M\to 0^+} \int_M^1 \frac{1}{\sqrt{x}} dx &= \lim_{M\to 0^+} 2\sqrt{x} \Big|_M^1 \\
&= \lim_{M\to 0^+} 2 - 2\sqrt{M} \\
&= \boxed{2}.
\end{aligned}$$

Note that this integral converges as well.

(2) **Two improper bounds or discontinuity inside the interval**: With every other improper integral, the goal will be to turn it into a bunch of integrals of the previous case.

Example 12.6. *Evaluate*

$$\int_{-1}^1 \frac{1}{x} dx.$$

Solution. It is only improper because of the discontinuity at $x = 0$, but it happens inside the interval. You make it happen at the bound by splitting up the integral (at the discontinuity):

$$\int_{-1}^{1} \frac{1}{x} dx = \int_{-1}^{0} \frac{1}{x} dx + \int_{0}^{1} \frac{1}{x} dx.$$

Now each integral is improper for exactly one reason, and that reason is one of the bounds. Therefore you evaluate each integral as we did in the previous case:

$$\begin{aligned}
\int_0^1 \frac{1}{x} dx &= \lim_{M \to 0^+} \int_M^1 \frac{1}{x} dx \\
&= \lim_{M \to 0^+} \ln|x| \Big|_M^1 \\
&= \lim_{M \to 0^+} \ln|1| - \ln|M| \\
&= \lim_{M \to 0^+} -\ln|M| \\
&= \infty,
\end{aligned}$$

so this integral diverges. As soon as one of the integrals of your sum diverges, the whole thing diverges, so we don't even need to compute the other integral $\int_{-1}^{0} \frac{1}{x} dx$. The original integral only converges if all your new integrals converge.

Example 12.7. *Evaluate*

$$\int_0^\infty \frac{1}{x(x+1)} dx.$$

Solution. This integral is improper because of the infinite bound, but it is also improper because of the discontinuity at $x = 0$ (the discontinuity at $x = -1$ does not matter because it is not in the interval $[0, \infty)$). We have two improper bounds, so we must split up the integral. Pick your favorite point between 0 and ∞:

$$\int_0^\infty \frac{1}{x(x+1)} dx = \int_0^1 \frac{1}{x(x+1)} dx + \int_1^\infty \frac{1}{x(x+1)} dx.$$

Now each integral has only one improper bound and nothing bad happens inside the interval, so we can evaluate each as above. We can write:

$$\int_0^1 \frac{1}{x(x+1)} dx = \lim_{M \to 0^+} \int_M^1 \frac{1}{x(x+1)} dx.$$

This is a partial fractions integral, and we leave it to the reader to find

$$\int_M^1 \frac{1}{x(x+1)} dx = \int_M^1 \frac{1}{x} - \frac{1}{x+1} dx = \ln|x| - \ln|x+1| \Big|_M^1 = [\ln|M| - \ln|M+1|] - [0 - \ln|2|].$$

We need to take a limit:

$$\lim_{M \to 0^+} \ln|M| - \ln|M+1| + \ln|2| = \ln|2| + \lim_{M \to 0^+} \ln|M| - \ln|M+1|$$

As $M \to 0^+$, $\ln|M| \to -\infty$, and $\ln|M+1| \to \ln(1) = 0$. So the whole limit approaches $-\infty$, and so the integral $\boxed{\text{diverges}}$.

Example 12.8. *Evaluate*

$$\int_{-\infty}^{\infty} \frac{x}{x^2+1} dx.$$

Solution. This has no discontinuities, but it has two infinite bounds, so we need to break up the integral:

$$\int_{-\infty}^{0} \frac{x}{x^2+1}dx + \int_{0}^{\infty} \frac{x}{x^2+1}dx.$$

Let's do the second integral first:

$$\int_{0}^{\infty} \frac{x}{x^2+1}dx = \lim_{M\to\infty} \int_{0}^{M} \frac{x}{x^2+1}dx.$$

Now,

$$\int_{0}^{M} \frac{x}{x^2+1}dx$$

requires a u-sub: $u = x^2+1$, $du = 2xdx$, so $xdx = \frac{du}{2}$. The $x = 0$ bound becomes a $u = 0^2+1 = 1$ bound, and the $x = M$ bound becomes $u = M^2+1$. So the integral is

$$\frac{1}{2}\int_{1}^{M^2+1} \frac{du}{u} = \frac{1}{2}\ln|u|\Big|_{1}^{M^2+1} = \frac{1}{2}[\ln|M^2+1| - \ln|1|] = \frac{1}{2}\ln|M^2+1|.$$

So

$$\lim_{M\to\infty} \int_{0}^{M} \frac{x}{x^2+1}dx = \lim_{M\to\infty} \frac{1}{2}\ln|M^2+1| = \infty,$$

so the whole integral diverges.

Example 12.9. *Evaluate* $\int_{-\infty}^{\infty} \frac{1}{x^2+1}dx$.

Solution. Again, there are no discotinuities, but we have two improper bounds, so we can break the integral up:

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}dx = \int_{-\infty}^{0} \frac{1}{x^2+1}dx + \int_{0}^{\infty} \frac{1}{x^2+1}dx.$$

The second integral we computed in Example 2.25, and it converged to $\pi/2$. So we also have to check the other integral:

$$\int_{-\infty}^{0} \frac{1}{x^2+1}dx = \lim_{N\to-\infty} \int_{N}^{0} \frac{1}{x^2+1}dx = \lim_{N\to-\infty} \tan^{-1}(x)\Big|_{N}^{0} = \lim_{N\to-\infty} \tan^{-1}(0) - \tan^{-1}(N).$$

Again, $\tan^{-1}(0) = 0$, and $\tan^{-1}(N) \to -\pi/2$ as $N \to -\infty$, so this integral also converges to $\pi/2$ (since $0 - (-\pi/2) = \pi/2$). Therefore our original integral converges, and

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}dx = \int_{\infty}^{0} \frac{1}{x^2+1}dx + \int_{0}^{\infty} \frac{1}{x^2+1}dx = \frac{\pi}{2} + \frac{\pi}{2} = \boxed{\pi}.$$

CAUTION. Be aware of the following:

(1) $\lim_{x\to\pm\infty} \sin(x)$ does not exist. The same goes for cosine and tangent.

(2) Please remember that

$$\int_{-\infty}^{\infty} f(x)dx \neq \lim_{M\to\infty} \int_{-M}^{M} f(x)dx.$$

You should know, now, that since this has at least two improper traits (two infinite bounds), we need to break up the integral.

(3) Don't forget $\infty - \infty$ and $0 \cdot \infty$ are indeterminate forms, so you should remember how to deal with these limits (look back at your Math 3A notes).

12.2 Comparison Test

Theorem 12.10. *Suppose $f(x)$ and $g(x)$ are continuous on $[a, \infty)$ with $0 \leq f(x) \leq g(x)$. Then*

$$0 \leq \int_a^\infty f(x)dx \leq \int_a^\infty g(x)dx.$$

The test says that if $\int_a^\infty g(x)dx$ converges, then so does $\int_a^\infty f(x)dx$, and if $\int_a^\infty f(x)dx$ diverges then so does $\int_a^\infty g(x)dx$. This should make sense: if $g(x)$ has finite area, then so must $f(x)$ since $f(x)$ is smaller, and if $f(x)$ has infinite area, then so does $g(x)$ since $g(x)$ is bigger.

In truth, we can apply this test to all sorts of improper integrals provided that there is only one improper bound and it happens at an endpoint. However, the functions *must be greater than or equal to zero.*

12.2.1 What to compare to: the p-test

Notice that, from the statement of the theorem, if we want to prove an integral converges, we need to find a bigger function which we know converges. Similarly, if we want to prove an integral diverges, then we must find a smaller function which we know diverges. This means that when you first encounter an integral, you have to make an educated guess as to whether you think the integral converges or diverges, because this determines which way you need to bound the function. To make these guesses, you want to keep the following in mind:

(1) Integrals of the form

$$\int_a^\infty \frac{1}{x^p}dx,$$

where $a > 0$ (must be strictly positive), converge when $p > 1$ and diverge when $p \leq 1$.

(2) Integrals of the form

$$\int_0^a \frac{1}{x^p}dx,$$

where $a > 0$ is not infinity, converge when $p < 1$ and diverge when $p \geq 1$. Notice nothing good ever happens with $p = 1$, and that everything here is the

(3) Now consider

$$\int_a^\infty e^{kx}dx,$$

where a is not $-\infty$. This converges if $k < 0$ and diverges if $k \geq 0$. Exponential decay is as good as it gets.

(4) Similarly, if we consider

$$\int_{-\infty}^a e^{kx}dx,$$

now with a not ∞, this converges if $k > 0$ and diverges if $k \leq 0$. With these last two, notice that all you need is for the graph of the exponential to be tending to 0 as you approach the infinite bound. That should make it easy to remember.

12.2.2 Examples

Example 12.11. *Determine whether $\int_1^\infty \frac{1}{x^4+1}dx$ converges or diverges.*

Solution. This is not an integral we know how to compute, so we use the comparison test. First, we need to make a guess, as this will determine which way we want inequalities to go. Notice that as $x \to \infty$, the bottom just behaves like x^4, since the extra 1 is insignificant. So the fraction behaves like $\frac{1}{x^4}$, and $\int_1^\infty \frac{1}{x^4} dx$ converges by case (1) of the integrals above. So we can make the guess that the integral converges. One thing you should try and do is "remove" the terms you don't care about, assuming the inequality goes in the proper direction when you do. In this case, we'd like to remove the 1 from the denominator, so is

$$\int_1^\infty \frac{1}{x^4+1} dx \leq \int_1^\infty \frac{1}{x^4} dx?$$

Well yes, since $x^4 + 1 \geq x^4$, meaning $\frac{1}{x^4+1} \leq \frac{1}{x^4}$, which is what we wanted. The other way to think about it is that the denominator of $\frac{1}{x^4+1}$ is bigger than that of $\frac{1}{x^4}$, so the fraction is smaller. The integral $\int_1^\infty \frac{1}{x^4} dx$ converges by the p-test, so by comparison, the original integral converges.

Example 12.12. *Determine whether*

$$\int_3^\infty \frac{\sin^2(x)}{x^4} dx$$

converges or diverges.

Solution. This is not something you want to try and integrate, but let's determine whether it converges or diverges. Now as x gets big, the $\sin^2(x)$ is not so significant (it is between 0 and 1), and the bottom is just x^4. Looking at the case (1) above again, $p = 4 > 1$, so we guess this should converge. To prove it, we need to find a bigger function which converges. Again, we'd like to remove the term we don't care about, this one being $\sin^2(x)$. We know $-1 \leq \sin(x) \leq 1$, so $0 \leq \sin^2(x) \leq 1$. So we have

$$\frac{\sin^2(x)}{x^4} \leq \frac{1}{x^4}.$$

Therefore

$$\int_3^\infty \frac{\sin^2(x)}{x^4} dx \leq \int_3^\infty \frac{1}{x^4} dx,$$

and since $\int_3^\infty \frac{1}{x^4} dx$ converges, so does our integral.

Example 12.13. *Determine whether*

$$\int_5^\infty \frac{x^2+x+1}{x^4+x^2+3} dx$$

converges or diverges.

Solution. Again, let's look at behavior. As $x \to \infty$, the top behaves like x^2 and the bottom like x^4, so the fraction behaves like $\frac{x^2}{x^4} = \frac{1}{x^2}$. Since $\int_5^\infty \frac{1}{x^2} dx$ converges, we again make the guess that it will converge. Now we need to find a good function to compare it to. This time, in the denominator, we do not need the x^2 and the 3, since they were insignificant compared to the x^4. Thankfully, removing them from the denominator makes it smaller, which makes the entire fraction bigger. So

$$\int_5^\infty \frac{x^2+x+1}{x^4+x^2+3} dx \leq \int_5^\infty \frac{x^2+x+1}{x^4} dx.$$

We cannot, however, remove the x and the 1 from the numerator, since this would make the numerator smaller and, consequently, the entire fraction smaller. In this example, there are two ways to go. The trick we wanted to illustrate here was that if you cannot remove terms as you'd like, then try and replace them by terms you do care about. In this case, we want x^2's in the numerator. If $x > 1$ (which it is since we are considering $[5, \infty)$), then $x \leq x^2$ and $1 \leq x^2$, so

$$\frac{x^2+x+1}{x^4} \leq \frac{x^2+x^2+x^2}{x^4} = \frac{3x^2}{x^4}.$$

Since $\int_5^\infty \frac{3}{x^2} dx$ converges, so does our integral.

Remark. We make two remarks:

(1) In this problem, we did not actually have to replace the numerator by $3x^2$. Instead, we could have observed

$$\int_5^\infty \frac{x^2+x+1}{x^4}dx = \int_5^\infty \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4}dx,$$

and each term converges by the p-test.

(2) If the integral was on the interval $[1/2, \infty)$, then we could not do the trick for the numerator since it required $x > 1$. However, when inequalities only work on part of your interval (more specifically, on some subinterval containing the improper part), then we could break up the integral:

$$\int_{1/2}^\infty \frac{x^2+x+1}{x^4+x^2+3}dx = \int_{1/2}^1 \frac{x^2+x+1}{x^4+x^2+3}dx + \int_1^\infty \frac{x^2+x+1}{x^4+x^2+3}dx.$$

The 1 was chosen as the cutoff because we needed $x > 1$. The second integral now converges by the work above. And for the first, you simply observe that it is not improper for any reason, and so it is just a number. Some number plus a convergent integral is convergent.

Example 12.14. *Determine whether* $\int_1^\infty \frac{1}{\sqrt{x}-1}dx$ *converges or diverges.*

Solution. Now, the behavior is like $\frac{1}{\sqrt{x}}$, and $\int_0^1 \frac{1}{\sqrt{x}}dx$ diverges (p-test and $p = 1/2 < 1$), so we make the guess that our integral diverges. This time, we want to find a smaller function which we know diverges. But notice that $\frac{1}{\sqrt{x}-1} \geq \frac{1}{\sqrt{x}}$ since the denominator of $\frac{1}{\sqrt{x}-1}$ is less than that of $\frac{1}{\sqrt{x}}$, which makes the fraction bigger. Alternatively, start with $\sqrt{x} - 1 \leq \sqrt{x}$ (which it is), and take reciprocals: $\frac{1}{\sqrt{x}-1} \geq \frac{1}{\sqrt{x}}$. So

$$\int_1^\infty \frac{1}{\sqrt{x}-1}dx \geq \int_1^\infty \frac{1}{\sqrt{x}}dx,$$

and since the latter integral diverges, so does our original integral.

Example 12.15. *Determine whether*

$$\int_1^\infty \frac{\ln^2(x)}{\sqrt{x}}dx$$

converges or diverges.

Solution. This is sort of as tricky as it gets. Now as $x \to \infty$ the $\ln^2(x)$ grows incredibly slowly, so the $\sqrt{x}$ dominates the fraction. Since $\int_1^\infty \frac{1}{\sqrt{x}}dx$ diverges, we can guess that our integral will too. So now we need to bound our function from *below*. But let's think: it would be nice if we could say $\ln^2(x) \geq 1$, since then

$$\frac{\ln^2(x)}{\sqrt{x}} \geq \frac{1}{\sqrt{x}},$$

and because $\int_1^\infty \frac{1}{\sqrt{x}}dx$, we would be done by the comparison test. However, this is not true on our interval. However, if $x \geq e$, then $\ln(x) \geq 1$, so $\ln^2(x) \geq 1$ as well. So instead, what we do is break up our integral:

$$\int_1^\infty \frac{\ln^2(x)}{\sqrt{x}}dx = \int_1^e \frac{\ln^2(x)}{\sqrt{x}}dx + \int_e^\infty \frac{\ln^2(x)}{\sqrt{x}}dx.$$

This first integral does not matter to us, since it is just a number (indeed, the function is continuous and the interval is finite), so it does not influence the convergence or divergence of our original integral. And for the second integral, we now use the bound we want: $\ln^2(x) \geq 1$, so

$$\int_e^\infty \frac{\ln^2(x)}{\sqrt{x}}dx \geq \int_e^\infty \frac{1}{\sqrt{x}}dx,$$

and this integral diverges by the p-test above. Therefore our integral diverges.

Example 12.16. *Determine whether $\int_1^\infty \frac{1}{x+e^x}dx$ converges or diverges.*

Solution. As x gets large, the e^x is much bigger than the x, so the function behaves like $\frac{1}{e^x} = e^{-x}$, and $\int_1^\infty e^{-x}dx$ converges. So we guess our integral converges. Indeed, we observe that

$$\int_1^\infty \frac{1}{x+e^x}dx \leq \int_1^\infty \frac{1}{e^x}dx$$

since removing the x makes our denominator smaller, thereby making the fraction bigger. Since the second integral converges, so does our initial integral.

Remark. Observe that we could have also said $\frac{1}{x+e^x} \leq \frac{1}{x}$, but this would not help us since the integral of $1/x$ would diverge. This is why it is generally good to keep track of the terms that influence the convergence or divergence of the integral when you make the initial guess.

Example 12.17. *Determine whether $\int_1^\infty e^{-x}e^{-1/x}$ converges or diverges.*

Solution. First, you see that this function is the same as $\frac{1}{e^x} \cdot \frac{1}{e^{1/x}}$. Now the $1/e^x \to 0$, but the $1/e^{1/x} \to 1$ as $x \to \infty$, not 0. So ti is the $\frac{1}{e^x}$ which we want to focus on. Since $\int_1^\infty \frac{1}{e^x}dx$ converges, we guess that our integral does as well. But finding the function to compare to is tougher. But you notice that if $1 \leq x < \infty$, then $e^{-1/x} < 1$, since e to any negative exponent is less than 1, and $-1/x$ is indeed negative for $x > 0$. So $e^{-x}e^{-1/x} < e^{-x}$, and thus

$$\int_1^\infty e^{-x}e^{-1/x}dx \leq \int_1^\infty e^{-x}dx,$$

and since the latter integral converges, so does our original integral.

13 Trapezoidal Rule and Error Bound (8.8)

Summary 13.1. *(1) To approximate $\int_a^b f(x)dx$ using trapezoidal rule with N subintervals, partition $[a, b]$ into N pieces, each piece having length $\Delta x = \frac{b-a}{N}$. Then*

$$T_N = \frac{1}{2} \cdot \frac{b-a}{N}(f(a) + 2f(a+\Delta x) + 2f(a+2\Delta x) + \ldots + 2f(a+(N-1)\Delta x) + f(a+N\Delta x)).$$

(2) If $f''(x)$ exists and continuous, and K_2 is any number with $|f''(x)| \leq K_2$ for any x in the interval $[a, b]$, then the error from T_N is at most $\frac{K_2(b-a)^3}{12N^2}$.

Example 13.2. *Approximate $\int_0^4 x^2dx$ using Trapezoidal rule with 4 trapezoids. Then find N so that the error from T_N is at most .001.*

Solution. First, $\Delta x = \frac{b-a}{N} = \frac{4-0}{4} = 1$. Using the formula,

$$T_4 = \frac{1}{2} \cdot 1(f(0) + 2f(1) + 2f(2) + 2f(3) + f(4)) = \frac{1}{2}(0 + 2 \cdot 1^2 + 2 \cdot 2^2 + 2 \cdot 3^2 + 4^2) = \boxed{22}.$$

To find N so that the error is at most .001, first find $f''(x)$: $f''(x) = 2$. We need to find K_2 so that $|f''(x)| \leq K_2$ on $[0, 4]$. But as $f''(x)$ is constant, we could just take $K_2 = 2$ (or any bigger number). So the error of T_N is at most $\frac{2(4-0)^3}{12N^2}$. Since we want the error to be at most .001, we set

$$\frac{2 \cdot 64}{12N^2} \leq .001 \implies \frac{128}{12 \cdot .001} \leq N^2 \implies N \geq \boxed{\sqrt{\frac{128}{12 \cdot .001}}}.$$

Before we start the next example, we will need a fact (which you are free to use): $|a + b| \leq |a| + |b|$. This is called the *triangle inequality.*

Example 13.3. *Find N so that the error from T_N to approximate $\int_0^\pi x\cos(x)dx$ is at most .01.*

Solution. Again, we just need K_2. If $f(x) = x\cos(x)$, calculate $f''(x) = -2\sin(x) - x\cos(x)$. Now, you do not want to actually calculate the maximum of $|f''(x)|$ on $[0,\pi]$. So instead, we will not try and find the *best* possible bound. Instead, we use the triangle inequality from above:

$$|-2\sin(x) - x\cos(x)| = |-2\sin(x) + -x\cos(x)| \le |-2\sin(x)| + |-x\cos(x)| = 2|\sin(x)| + |x\cos(x)|.$$

Now, the maximum of $\sin(x)$ on $[0,\pi]$ is 1, and the same for $\cos(x)$. Finally, since $|x|$ can be at most π on $[0,\pi]$, we see

$$2|\sin(x)| + |x\cos(x)| \le 2\cdot 1 + \pi\cdot 1 = 2+\pi.$$

So we will let $K_2 = 2+\pi$. So the error is at most $\frac{K_2(b-a)^3}{12N^2} = \frac{(2+\pi)(\pi-0)^3}{12N^2}$. To find N, we set

$$\frac{(2+\pi)\pi^3}{12N^2} \le .01 \implies \frac{(2+\pi)\pi^3}{12(.01)} \le N^2 \implies N \ge \boxed{\sqrt{\frac{(2+\pi)\pi^3}{12(.01)}}}.$$

Example 13.4. *Find N so that the error from T_N to approximate $\int_1^2 \frac{1}{x}dx$ is at most* .01.

Solution. Again, $f(x) = \frac{1}{x}$ means $f''(x) = \frac{2}{x^3}$. Now, $|f''(x)|$ will be biggest when x is smallest (since smaller denominators mean bigger fractions), so in this case, when $x = 1$. So $|f''(x)| \le \frac{2}{1^3} = 2$. So $K_2 = 2$ again. So

$$\frac{2(2-1)^3}{12N^2} \le .01 \implies \frac{2}{12(.01)} \le N^2 \implies N \ge \boxed{\sqrt{\frac{2}{12(.01)}}}.$$

14 Sequences (11.1)

First, we list the main ideas, and then we'll elaborate and illustrate.

Summary 14.1. *(1) A sequence a_n has a limit L if a_n "gets really close" to L as $n \to \infty$*

(2) Let $a_n = f(n)$ be a sequence. If $f(x) \to L$ as $x \to \infty$, then $a_n \to L$.

(3) If $f(x)$ is continuous and $a_n \to L$, then $f(a_n) \to f(L)$.

(4) If $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences with $b_n \le a_n \le c_n$ (for n large enough), and b_n and c_n both have limit L, then $a_n \to L$ as well.

(5) If $\{a_n\}$ is an increasing sequence which is bounded above, then a_n converges. Similarly, if $\{a_n\}$ is a decreasing sequence which is bounded below, then a_n converges.

(6) Formally, a sequence $a_n \to L$ if for all $\epsilon > 0$, there exists M such that for all $n > M$, $|a_n - L| < \epsilon$.

Intuitively, the limit of a sequence is the number the sequence gets close to as $n \to \infty$, if one exists. The easiest example would probably be $a_n = \frac{1}{n}$. This has terms $1, 1/2, 1/3, \ldots$ (corresponding to $n = 1, 2, 3, \ldots$) and, over time, this sequence of numbers gets really close to 0. So a_n has limit 0, and it *converges.*

On the other hand, the sequence $a_n = (-1)^n$ has terms $-1, 1, -1, 1, \ldots$, and so it does not approach a single number. Therefore no limit exists, and the sequence *diverges.*

Example 14.2. *Find the limit of $a_n = \frac{n+1}{2n}$.*

Solution. Most of the time, we will use item (2) above. What is says is that instead of looking at $\frac{n+1}{2n}$, we can look at $f(x) = \frac{x+1}{2x}$. This is good because now we can use calculus. In particular, we can use L'Hopital:

$$\lim_{x\to\infty} \frac{x+1}{2x} \overset{\text{LH}}{=} \lim_{x\to\infty} \frac{1}{2} = \frac{1}{2}.$$

Since we got a limit here, the limit of a_n is also $\boxed{\frac{1}{2}}$.

Example 14.3. *Find the limit of* $a_n = e^{-n}\cos\left(\frac{1}{n}\right)$.

Solution. Again, we can let $f(x) = e^{-x}\cos\left(\frac{1}{x}\right)$. Now, as $x \to \infty$, $e^{-x} \to 0$, and $\cos(1/x) \to 1$ since $1/x \to 0$ (and $\cos(0) = 1$). Therefore, the limit of $f(x)$ is 0, meaning the limit of a_n is also $\boxed{0}$.

Example 14.4. *Find the limit of* $a_n = \sin(n\pi)$.

Solution. If $f(x) = \sin(x\pi)$, then $f(x)$ has no limit as $x \to \infty$ (as the graph oscialltes). However, we cannot conclude that the sequence diverges. In fact, the sequence does have a limit. If we list out the first few terms, then we get $0, 0, 0, \ldots$ (since $0 = \sin(\pi) = \sin(2\pi) = \sin(3\pi) = \ldots$). Therefore the limit of this sequence is just $\boxed{0}$.

CAUTION. The previous example illustrates an important point, which is that item (2) in the summary is NOT applicable if $f(x)$ has no limit as $x \to \infty$. You must try something else in such situations (such as listing the first few terms to get a feel for what is happening). This example also shows that while $\sin(n)$ has no limit as $n \to \infty$, the π inside the sin makes things work. The reader should check that the sequence $a_n = \sin(n\pi/2)$ also has no limit.

Example 14.5. *Find the limit of* $a_n = e^{\frac{n+1}{2n^2+3}}$

Solution. This is one of those situations where item (3) from the summary is useful. Since $f(x) = e^x$ is continuous, it suffices to see where the exponent $\frac{n+1}{2n^2+3}$ goes as $n \to \infty$. For this, let $g(x) = \frac{x+1}{2x^2+3}$. Then as $x \to \infty$, we have ∞/∞, so we can use L'Hopital's rule:

$$\lim_{x\to\infty} \frac{x+1}{2x^2+3} = \lim_{x\to\infty} \frac{1}{4x} = 0.$$

Therefore $\frac{n+1}{2n^2+3} \to 0$, and so $a_n \to e^0 = \boxed{1}$.

Example 14.6. *Find the limit of* $a_n = \ln\left(\frac{1}{n}\right)$.

Solution. As $n \to \infty$, $\frac{1}{n} \to 0$. So we have ln of something approaching 0, to this sequences diverges to $-\infty$.

Example 14.7. *Find the limit of* $a_n = \frac{\sin(n)}{n}$

Solution. Here, as $n \to \infty$, we have sin of something going to ∞, which is a big clue for squeeze theorem (as is cos of something going to ∞). To use it, we can say

$$-1 \leq \sin(n) \leq 1,$$

so

$$\frac{-1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}.$$

Since both $\pm 1/n \to 0$, we get our limit is $\boxed{0}$.

Example 14.8. *Find the limit of* $a_n = \frac{(-1)^n}{2n+3}$

Solution. Again, as $n \to \infty$, the $(-1)^n$ is bouncing between -1 and 1. This is also another classic squeeze theorem problem. Here, $(-1)^n$ is between -1 and 1 (in fact, it is always one of these two):

$$-1 \leq (-1)^n \leq 1 \implies \frac{-1}{2n+3} \leq \frac{(-1)^n}{2n+3} \leq \frac{1}{2n+3}.$$

Since both $\pm\frac{1}{2n+3} \to 0$ as $n \to \infty$, our limit is again $\boxed{0}$.

Remark. The previous two examples show that you typically use squeeze theorem when we have these "osciallating" terms as $n \to \infty$ which we want to get rid of to make the limit easier. Note that we would not do this in the case $a_n = \sin(n)$, since this limit does not exist. The only reason the previous two examples worked is because we had other terms making the whole thing go to 0.

Example 14.9. *Find the limit of* $a_n = \frac{\sin(n)-\cos(n)}{e^n}$.

Solution. This is again a squeeze theorem problem. Now the problem is the $\sin(n)-\cos(n)$ (as both of these have no limits as $n \to \infty$), and so this is what we try and bound. Since $\sin(n)$ and $\cos(n)$ are both bounded by ± 1, we can say

$$-2 \leq \sin(n) - \cos(n) \leq 2.$$

(The biggest $\sin(n)-\cos(n)$ could be is when $\sin(n) = 1$ but $\cos(n) = -1$, even if this never actually happens. Similarly for the lower bound.) So

$$\frac{-2}{e^n} \leq a_n \leq \frac{2}{e^n}.$$

Since both $\frac{\pm 2}{e^n} \to 0$, we get $a_n \to \boxed{0}$.

A sequence $\{a_n\}$ is *bounded above* if there is a number B such that $a_n < B$ for all n. Similarly, a sequence $\{a_n\}$ is *bounded below* if there exists B such that $a_n > B$ for all n. Finally, a sequence is *bounded* if there exists A and B with $A < a_n < B$ for all n (i.e. it is bounded both above and below). We have the following theorem:

Theorem 14.10. *An increasing sequence which is bounded above converges. Similarly, a decreasing sequence which is bounded below converges.*

Example 14.11. *Let* $a_1 = 2$ *and* $a_{n+1} = \sqrt{2a_n + 1}$. *Show:*

(a) $a_n < 4$ *for all* n.

(b) $a_{n+1} > a_n$ *for all* n.

(c) Conclude that a_n *converges, and find the limit.*

Solution. (a) To show $a_n < 4$ we use induction:

(i) Base case: $n = 1$: We have $a_1 = 2 < 4$, which means the base case holds.

(ii) Assume it is true for $n = k$, we want to show it is true for $n = k + 1$: Being true for $n = k$ means $a_k < 4$. We want to show $a_{k+1} < 4$. We know $a_{k+1} = \sqrt{2a_k + 1}$. Since $a_k < 4$ by assumption, $a_{k+1} = \sqrt{2a_k + 1} < \sqrt{2 \cdot 4 + 1} = 3 < 4$. Therefore the inductive step also holds, and the statement is true for all n.

(b) Again, to show $a_{n+1} > a_n$, we use induction:

(i) Base case: $n = 1$: We need $a_2 > a_1$. We know $a_2 = \sqrt{2a_1 + 1} = \sqrt{2(2) + 1} = \sqrt{5}$, and $a_1 = 2$. Since $\sqrt{5} > 2$, we know $a_2 > a_1$, and the base case holds.

(ii) Assume it is true for $n = k$, we want to show it is true for $n = k + 1$: Being true for $n = k$ means $a_{k+1} > a_k$. We want to show $a_{k+2} > a_{k+1}$, i.e. $\sqrt{2a_{k+1} + 1} > \sqrt{2a_k + 1}$. We know $a_{k+2} = \sqrt{2a_{k+1} + 1}$, and since $a_{k+1} > a_k$, $\sqrt{2a_{k+1} + 1} > \sqrt{2a_k + 1} = a_{k+1}$, as desired. Therefore the inductive step holds and the statement is true for all n.

(c) Since $\{a_n\}$ is an increasing sequence (by (b)) which is bounded above (by (a)), we know the sequence converges. To find the limit, say $a_n \to L$. Then $a_{n+1} \to L$ as well (the shift doesn't change the limit). So taking limit as $n \to \infty$ of both sides of $a_{n+1} = \sqrt{2a_n + 1}$ gives $L = \sqrt{2L + 1}$, so $L^2 = 2L + 1$. To solve $L^2 - 2L - 1 = 0$, we use the quadratic formula to get $L = 1 \pm \sqrt{2}$. Since $a_n > 0$ for all n, we have to take the positive root, so the limit is $1 + \sqrt{2}$.

Example 14.12. *Let* $a_1 = 2$ *and* $a_{n+1} = \frac{1}{4}(a_n + 5)$. *Show:*

(a) $a_{n+1} < a_n$

(b) $a_n > 1$ *for all* n.

(c) Conclude that $\{a_n\}$ *converges, and find the limit.*

Solution. (a) To show $a_{n+1} < a_n$, we use induction:

(i) Base case: $n = 1$: We need $a_2 < a_1$. We know $a_2 = \frac{1}{4}(a_1 + 5) = \frac{1}{4}(2 + 5) = \frac{7}{4}$, and $a_1 = 2$. Since $\frac{7}{4} < 2$, we know $a_2 < a_1$, and the base case holds.

(ii) Assume it is true for $n = k$, we want to show it is true for $n = k + 1$: Being true for $n = k$ means $a_{k+1} < a_k$. We want to show $a_{k+2} < a_{k+1}$, i.e. $\frac{1}{4}(a_{k+1} + 5) < \frac{1}{4}(a_k + 5)$. We know $a_{k+2} = \frac{1}{4}(a_{k+1} + 5)$, and since $a_{k+1} < a_k$, $\frac{1}{4}(a_{k+1} + 5) < \frac{1}{4}(a_k + 5) = a_{k+1}$, as desired. Therefore the inductive step holds and the statement is true for all n.

(b) To show $a_n > 1$ we use induction:

(i) Base case: $n = 1$: We have $a_1 = 2 > 1$, which means the base case holds.

(ii) Assume it is true for $n = k$, we want to show it is true for $n = k+1$: Being true for $n = k$ means $a_k > 1$. We want to show $a_{k+1} > 1$. We know $a_{k+1} = \frac{1}{4}(a_k + 5)$. Since $a_k > 1$ by assumption, $a_{k+1} = \frac{1}{4}(a_k + 5) > \frac{1}{4}(1+5) = \frac{3}{2} > 1$. Therefore the inductive step also holds, and the statement is true for all n.

(c) Since $\{a_n\}$ is bounded below by (b) and decreasing by (a), it converges. To find the limit, say $a_n \to L$. Again, $a_{n+1} \to L$ as well. Taking the limit as $n \to \infty$ of both sides of $a_{n+1} = \frac{1}{4}(a_n + 1)$, we get $L = \frac{1}{4}(L + 5)$. Solving for L gives $L = \frac{5}{3}$.

The last thing we will do it use the ϵ-definition of a limit of a sequence. All item (6) of the summary is saying that for any ϵ, which you should think of an "error," if we go out far enough in the sequence, then the terms of the sequence are within this error of the limit of the sequence.

Example 14.13. *Find M so that $|a_n - 1| < .01$ for $n > M$, where $a_n = \frac{n^2}{n^2+1}$.*

Solution. We want to find M so that $\left|\frac{n^2}{n^2+1} - 1\right| < .01$. We can just try solving this inequality for n: Combining the fractions yields:

$$\left|\frac{n^2}{n^2+1} - 1\right| < .01 \implies \left|\frac{-1}{n^2+1}\right| < .01 \implies \frac{1}{n^2+1} < .01 \implies 1 < .01(n^2+1) \implies 99 < n^2 \implies \sqrt{99} < n.$$

So letting $M = \sqrt{99}$ will do it.

15 Summing Infinite Series (11.2)

Definition 15.1. *An infinite series is just an infinite sum, usually written like*

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \dots .$$

An example would be

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots ,$$

where the terms are gotten by plugging in $n = 1, 2, 3, \dots$ into $\frac{1}{n}$.

Definition 15.2. *Let $\sum_{n=1}^{\infty} a_n$ be a series. The N-th partial sum, denoted S_N, is*

$$S_N = \sum_{n=1}^{N} a_n = a_1 + a_2 + \dots + a_N .$$

Example 15.3. *For the series $\sum_{n=1}^{\infty} \frac{1}{n}$, write out the N-th partial sum S_N. Also write out S_5.*

Solution. The N-th partial sum would be

$$S_N = \sum_{n=1}^{N} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{N}.$$

So,

$$S_5 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}.$$

So partial sums are just what we get by stopping the series after N terms. To determine the sum of an infinite series, what we're going to do is instead consider the sequence of partials sums: $b_n = S_n$. So the sequence we are looking at is just $S_1, S_2, S_3, \ldots$. Again, for the $\sum_{n=1}^{\infty} \frac{1}{n}$ example, this would correspond to the sequence

$$1, 1 + \frac{1}{2}, 1 + \frac{1}{2} + \frac{1}{3}, 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}, \ldots$$

and so on.

Definition 15.4. *If the sequence $\{S_n\}$ of partial sums converges to a number S, then the series is said to converge, and the sum is $\sum_{n=1}^{\infty} a_n = S$. If the limit of the S_n does not exist, the series diverges.*

Example 15.5. *Sum the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$.*

Solution. We will see later that the N-th partial sum is

$$S_N = \sum_{n=1}^{N} \frac{1}{2^n} = \frac{2^N - 1}{2^N}.$$

So to determine the sum, we take

$$\lim_{N \to \infty} S_N = \lim_{N \to \infty} \frac{2^N - 1}{2^N}.$$

This is now a limit of a sequence. To do this, we can notice that by L'Hopital's rule,

$$\lim_{x \to \infty} \frac{2^x - 1}{2^x} \overset{\text{LH}}{=} \lim_{x \to \infty} \frac{2^x \ln(2)}{2^x \ln(2)} = 1.$$

Therefore $\lim_{N \to \infty} \frac{2^N - 1}{2^N} = 1$. (The alternative would have been simply to write $\frac{2^N - 1}{2^N} = 1 - \frac{1}{2^N}$ and notice that $1/2^N \to 0$.) Since $\lim_{N \to \infty} S_N = 1$, the sum

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

Example 15.6. *Determine whether the series $\sum_{n=1}^{\infty} \frac{2}{n(n+2)}$ converges or diverges. If it converges, find the sum.*

Solution. If we want to find the sum, we will need the partial sums. To do this, observe that

$$\frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2}.$$

(You could get this using partial fractions if you want.) So

$$\sum_{n=1}^{\infty} \frac{2}{n(n+2)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2} \right).$$

To find the partial sum, we just write out that

$$S_N = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \ldots + \left(\frac{1}{N} - \frac{1}{N+2}\right).$$

(This is just the first N terms of the series.) Not all the middle terms cancel. For example, the $\frac{1}{2}$ from the second term will not cancel. The terms remaining are

$$S_N = 1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2}.$$

As $N \to \infty$, $S_N \to \frac{3}{2}$, and so the sum

$$\sum_{n=1}^{\infty} \frac{2}{n(n+2)} = \boxed{\frac{3}{2}}.$$

In particular, the series converges.

Remark. The previous example is called a telescoping series. These come up frequently, so be sure you understand it.

One thing to notice immediately is the following. Suppose a_n does not go to 0. Then in the infinite sum $\sum_{n=1}^{\infty} a_n$, we have no hope of converging because the partial sums will just get bigger (or jump around if we have an alternating series). For example, if we consider

$$\sum_{n=1}^{\infty} \frac{n}{n+1},$$

the terms $a_n = \frac{n}{n+1}$ approach 1 as $n \to \infty$, and so we are essentially adding 1 infinitely many times. So the series must diverge. This is summarized by the following theorem.

Theorem 15.7. *If the terms a_n do not approach 0 as $n \to \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.*

CAUTION. This point cannot be stressed enough: If the a_n do go to 0, that does not tell you anything about the series. For example, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges even though $\frac{1}{n} \to 0$ as $n \to \infty$.

The last type of series from this section is a *geometric series*. This is a series of the form

$$\sum_{n=0}^{\infty} cr^n = c + cr + cr^2 + \dots.$$

What this means is to get from one term of the series to the next, we just multiply by r, called the *ratio*. An example would be

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \dots.$$

To get from one term to the next, we multiply by $r = 1/2$. Another example would be

$$\sum_{n=5}^{\infty} \frac{5}{3^n} = \frac{5}{3^5} + \frac{5}{3^6} + \frac{5}{3^7} + \dots,$$

where the ratio here is now $r = 1/3$.

The sums of geometric series are summarized in the following theorem:

Theorem 15.8. *(1) The finite sum*

$$\sum_{n=0}^{N} cr^n = c + cr + cr^2 + \dots = \frac{c(1 - r^{N+1})}{1 - r}.$$

(2) If $|r| < 1$, then

$$\sum_{n=0}^{\infty} cr^n = \frac{c}{1 - r}.$$

Remark. Note that the sum in (1) is the N-th partial sum of the series in (2), and that the sum in (2) is the limit as $N \to \infty$ of the sum in (1).

Remark. It is important that $|r| < 1$. If it is not, then the series diverges. Also, it is not crucially important that the sum starts at $n = 0$. The way to remember the sum in (2) is that it is the first term in the series divided by $1 - r$. We will see this in the examples.

Example 15.9. *Find the sum of* $\sum_{n=0}^{\infty} \frac{1}{3^n}$.

Solution. This is just the series

$$\sum_{n=0}^{\infty} \frac{1}{3^n} = 1 + \frac{1}{3} + \frac{1}{9} + \dots.$$

So the first term is 1 and the ratio is 1/3. The formula in the theorem above gives

$$\sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{1}{1 - \frac{1}{3}} = \boxed{\frac{3}{2}}.$$

Example 15.10. *Find the sum of* $\sum_{n=5}^{\infty} \left(\frac{2}{3}\right)^n$.

Solution. This is again geometric, but we start at $n = 5$, not $n = 0$. By listing out the terms, we see that

$$\sum_{n=5}^{\infty} \left(\frac{2}{3}\right)^n = \left(\frac{2}{3}\right)^5 + \left(\frac{2}{3}\right)^6 + \left(\frac{2}{3}\right)^7 + \dots,$$

and so our first term is $\left(\frac{2}{3}\right)^5$ and our ratio is 2/3. Since $2/3 < 1$ the series will converge, and the sum is

$$\frac{\left(\frac{2}{3}\right)^5}{1 - \frac{2}{3}} = \boxed{3\left(\frac{2}{3}\right)^5}.$$

Remark. You could make the series above start at $n = 0$ by shifting everything by 5. This means everywhere you see an n, replace it by $n + 5$:

$$\sum_{n=5}^{\infty} \left(\frac{2}{3}\right)^n = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^{n+5} = \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^5 \left(\frac{2}{3}\right)^n.$$

Now we see that it is in the form of the theorem, that $c = \left(\frac{2}{3}\right)^5$, and $r = \frac{2}{3}$. But remembering that it is just first term divided by $1 - r$ allows you to avoid such work if you want.

Example 15.11. *Find the sum of* $\sum_{n=1}^{\infty} \left(\frac{-1}{2}\right)^{2n+1}$.

Solution. This is still geometric because we see the number to a power involving n. We could find the first term and r by listing out the terms, but we can also do it algebraically. To find r, we need it only to the n, not $2n$, so write

$$\sum_{n=1}^{\infty} \left(\frac{-1}{2}\right)^{2n+1} = \sum_{n=1}^{\infty} \left(\frac{-1}{2}\right)^{2n} \cdot \left(\frac{-1}{2}\right)^1 = \sum_{n=1}^{\infty} \left[\left(\frac{-1}{2}\right)^2\right]^n \cdot \left(\frac{-1}{2}\right) = \sum_{n=1}^{\infty} \left(\frac{-1}{2}\right) \cdot \left(\frac{1}{4}\right)^n.$$

Now we see that $r = 1/4$. Our first term can be found by just plugging in $n = 1$ to get $-\frac{1}{2} \cdot \frac{1}{4} = -\frac{1}{8}$. Since $|r| < 1$, the series converges, and it is

$$\frac{\frac{-1}{8}}{1 - \frac{1}{4}} = \boxed{\frac{-1}{6}}.$$

Now let's look at examples from the entire section.

Example 15.12. *Determine whether the series* $\sum_{n=1}^{\infty} \frac{2^n}{n}$ *converges or not. If it does, find the sum.*

Solution. Whenever you see a series in this class, you really should check to see whether $a_n \to 0$. In this case,

$$\lim_{n\to\infty} \frac{2^n}{n} = \infty,$$

since $\lim_{x\to\infty} \frac{2^x}{x} = \infty$ by L'Hopital's rule. Therefore a_n does not go to 0, and the series $\boxed{\text{diverges}}$.

Example 15.13. *Determine whether the series $\sum_{n=-1}^{\infty} \frac{(-3)^n}{5^{2n-1}}$ converges or diverges. If it converges, find the sum.*

Solution. First, as $n \to \infty$, $\frac{(-3)^n}{5^{2n+1}} \to 0$, and so we cannot use that test. This looks like a geometric series, but we have multiple n's floating around. But we can write

$$\frac{(-3)^n}{5^{2n-1}} = \frac{(-3)^n}{5^{2n}5^{-1}} = \left(\frac{-3}{5^2}\right)^n \cdot \frac{1}{5^{-1}} = 5\left(\frac{-3}{25}\right)^n.$$

Now, in the sum

$$\sum_{n=-1}^{\infty} 5\left(\frac{-3}{25}\right)^n,$$

the first term is $5 \cdot \left(\frac{-3}{25}\right)^{-1}$ and the ratio is $-3/25$. Since $|-3/25| < 1$, the series converges, and the sum is

$$\boxed{\frac{5 \cdot \left(\frac{-3}{25}\right)^{-1}}{1 - \frac{-3}{25}}}.$$

Example 15.14. *Determine whether the series $\sum_{n=2}^{\infty} \pi^{1-4n}$ converges. If so, find the sum.*

Solution. As $n \to \infty$, $\pi^{1-4n} \to 0$ as the exponent is going to $-\infty$ and $\pi > 1$. So we cannot use the divergence test. This is also geometric, and since

$$\pi^{1-4n} = \pi \cdot \pi^{-4n} = \pi \cdot \frac{1}{\pi^{4n}} = \pi \cdot \left(\frac{1}{\pi^4}\right)^n,$$

we get that our ratio is $1/\pi^4 < 1$. Also, our first term is $\pi^{1-4(2)} = \pi^{-7}$. So the series converges and the sum is

$$\boxed{\frac{\pi^{-7}}{1 - \frac{1}{\pi^4}}}.$$

Example 15.15. *Determine whether the series $\sum_{n=1}^{\infty} \frac{5^{2n}}{4.99^n}$ converges or diverges. If it converges, find the sum.*

Solution. Here, as $n \to \infty$, the 5^{2n} will grow much faster than the 4.99^n, so the $a_n \to \infty$, not 0, so the series automatically diverges.

Example 15.16. *Determine whether the series $\sum_{n=2}^{\infty} \frac{2}{n^2-1}$ converges or diverges. If it converges, find the sum.*

Solution. As $n \to \infty$, $\frac{2}{n^2-1} \to 0$, so we cannot use that test. Also unfortunate is the fact that this is not a geometric series, since we don't see number to the n anywhere. But we notice we can factor the bottom as $(n+1)(n-1)$, and a partial fraction decomposition will show that

$$\frac{2}{n^2-1} = \frac{1}{n-1} - \frac{1}{n+1},$$

and so

$$\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \sum_{n=2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n+1}\right).$$

This will be a telescoping series. To sum it, we find the partial sum:

$$S_N = \sum_{n=2}^{N} \left(\frac{1}{n-1} - \frac{1}{n+1} \right) = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \ldots + \left(\frac{1}{N-2} - \frac{1}{N}\right) + \left(\frac{1}{N-1} - \frac{1}{N+1}\right).$$

Here, almost all the middle terms cancel, except the $1/2$ from the second term and the $-1/N$ from the second to last term (this would only be cancelled by the next term of the series, which is $\frac{1}{N} - \frac{1}{N+2}$.) So

$$S_N = 1 + \frac{1}{2} - \frac{1}{N} - \frac{1}{N+1},$$

and so $S_N \to \frac{3}{2}$ as $N \to \infty$. Therefore the sum converges to $\boxed{\frac{3}{2}}$.

Remark. If you are unsure about which terms cancel, just list out a few partial sums and see, like S_5 or S_6. If we write out S_5, we get

$$S_5 = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{7}\right).$$

Here a lot cancels, and it simplifies to

$$S_5 = 1 + \frac{1}{2} - \frac{1}{6} - \frac{1}{7}.$$

Similarly,

$$S_6 = \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \left(\frac{1}{6} - \frac{1}{8}\right),$$

and now the stuff left over is $1 + \frac{1}{2} - \frac{1}{7} - \frac{1}{8}$. So you see that everything cancels except the last two minus terms, which in general would be $-\frac{1}{N}$ and $-\frac{1}{N+1}$.

Example 15.17. *Determine whether the series* $\sum_2^\infty \frac{n}{\sqrt{4n^2-1}}$ *converges or diverges. If it converges, find the sum.*

Solution. First, we need $a_n \to 0$. So we want to calculate $\lim_{n\to\infty} \frac{n}{\sqrt{4n^2-1}}$. If we translate this to x's, we would get

$$\lim_{x\to\infty} \frac{x}{\sqrt{4x^2-1}}.$$

Now we can use L'Hopital (we have ∞/∞), and doing so gives

$$\lim_{x\to\infty} \frac{x}{\sqrt{4x^2-1}} \overset{\text{LH}}{=} \lim_{x\to\infty} \frac{1}{\frac{4x}{\sqrt{4x^2-1}}} = \lim_{x\to\infty} \frac{\sqrt{4x^2-1}}{4x}.$$

So L'Hopital's rule did not help at all, so we can try something else. The dominant term trick tells us that $\frac{n}{\sqrt{4n^2-1}}$ should go to $\frac{n}{\sqrt{4n^2}} = \frac{1}{2}$. To show this properly, we can factor out an n from the bottom:

$$\frac{n}{\sqrt{4n^2-1}} = \frac{n}{\sqrt{n^2\left(4 - \frac{1}{n^2}\right)}} = \frac{n}{n\sqrt{4 - \frac{1}{n^2}}} = \frac{1}{\sqrt{4 - \frac{1}{n^2}}} \to \frac{1}{2}$$

as $n \to \infty$. Since a_n does not go to 0, the series diverges.

Remark. The alternative method to find the limit of a_n above is to multiply the top and bottom by $\frac{1}{n}$ (since n is the dominant term):

$$\frac{n}{\sqrt{4n^2-1}} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \frac{1}{\frac{1}{n}\sqrt{4n^2-1}} = \frac{1}{\sqrt{\frac{1}{n^2}(4n^2-1)}} = \frac{1}{\sqrt{4 - \frac{1}{n^2}}} \to \frac{1}{2}$$

as $n \to \infty$.

16 Series with Positive Terms (11.3)

In this section, you learn four tests to determine the convergence or divergence of series. They are outlined in the summary below:

Summary 16.1. *(1)* ***p-test****: The series* $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *converges if* $p > 1$ *and diverges if* $p \leq 1$.

(2) ***Integral Test****: If* $a_n = f(n)$*, where* $f(x)$ *is positive, decreasing, and continuous for* $x \geq c$*, then* $\sum_{n=c}^{\infty} a_n$ *converges if and only if* $\int_a^{\infty} f(x)dx$ *converges. (This means the integral and series behave the same way in terms of convergence/divergence.)*

(3) ***Comparison Test****: If there is a number* M *so that* $0 \leq a_n < b_n$ *for* $n \geq M$*, then:*

(i) If $\sum_{n=1}^{\infty} b_n$ *converges, then so does* $\sum_{n=1}^{\infty} a_n$.

(ii) If $\sum_{n=1}^{\infty} a_n$ *diverges, then so does* $\sum_{n=1}^{\infty} b_n$.

(4) ***Limit Comparison Test****: Let* $\{a_n\}$ *and* $\{b_n\}$ *be positive sequences, and suppose* $L = \lim_{n\to\infty} \frac{a_n}{b_n}$ *exists. Then:*

(i) If $0 < L < \infty$*, then* $\sum_{n=1}^{\infty} a_n$ *converges if and only if* $\sum_{n=1}^{\infty} b_n$ *converges.*

(ii) If $L = \infty$ *and* $\sum_{n=1}^{\infty} a_n$ *converges, then so does* $\sum_{n=1}^{\infty} b_n$.

(iii) If $L = 0$ *and* $\sum_{n=1}^{\infty} b_n$ *converges, then so does* $\sum_{n=1}^{\infty} a_n$.

Remark. (1) In addition to the geometric series we encountered in the previous section, this puts our test count up to five.

(2) The comparison test for series is no different than that of integrals. The p-test is also the same as the $\int_1^{\infty} \frac{1}{x^p}dx$ p-test (this works because of the integral test).

Let us look at a few examples of each, then start looking at random series.

Example 16.2. *Use the integral test to determine whether the series* $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ *converges or diverges.*

Solution. The integral test says to look at $\int_2^{\infty} \frac{1}{x\ln(x)}dx = \lim_{M\to\infty} \int_2^{M} \frac{1}{x\ln(x)}dx$. (Notice the bottom bound is 2 because the first term of the series is $n = 2$.) We evaluate this using u-sub: $u = \ln(x)$, $du = \frac{1}{x}dx$, so

$$\int_2^M \frac{1}{x\ln(x)}dx = \int_{\ln(2)}^{\ln(M)} \frac{1}{u}du = \ln|u| \Big|_{\ln(2)}^{\ln(M)} = \ln|\ln(M)| - \ln|\ln(2)|.$$

Notice we used the substitution to change our bounds from the first to second integral. Taking a limit as $M \to \infty$, the first term goes off to ∞ and the second term is constant, so the integral diverges. Therefore the series $\boxed{\text{diverges}}$.

Example 16.3. *Use the integral test to determine whether the series* $\sum_{n=1}^{\infty} ne^{-n^2}$ *converges or diverges.*

Solution. Again, we look at the corresponding integral $\int_1^{\infty} xe^{-x^2}dx = \lim_{M\to\infty} \int_1^M xe^{-x^2}dx$. We again do a u-sub, with $u = -x^2$, so $du = -2xdx$, and $xdx = -\frac{1}{2}du$:

$$\int_1^M xe^{-x^2}dx = -\frac{1}{2}\int_{-1}^{-M^2} e^u du = -\frac{1}{2}(e^u) \Big|_{-1}^{-M^2} = -\frac{1}{2}(e^{-M^2} - e^{-1}).$$

As $M \to \infty$, $e^{-M^2} \to 0$, so the answer is $\frac{1}{2}e^{-1}$. In particular, the integral converges, and therefore the series $\boxed{\text{converges}}$.

Remark. We are not claiming that the series converges to $\frac{1}{2}e^{-1}$, all we can say is that the series converges.

Example 16.4. *Use both comparison tests to determine whether* $\sum_{n=1}^{\infty} \frac{1}{n^3+n}$ *converges or diverges.*

Solution. (1) **Direct Comparison**: As $n \to \infty$, the fraction behaves like $\frac{1}{n^3}$, and the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges by the p-test ($p = 3$). So we guess that it converges and we want to find something which is bigger. The usual trick is to try and get rid of the term we don't care about (in this case, the n), and we can because $n^3 + n \geq n^3$, meaning

$$\frac{1}{n^3+n} \leq \frac{1}{n^3}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, our original series $\boxed{\text{converges}}$.

(2) **Limit Comparison**: In this case, the first thing to do is to still look at behavior. It behaves like $1/n^3$, but now instead of going through the entire comparison, we just say: let $a_n = \frac{1}{n^3+n}$ (our original term) and let $b_n = \frac{1}{n^3}$ (the thing we want to compare it to). Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^3+n}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^3}{n^3+n}.$$

Dominant terms tell us that this limit should be 1. We could also use L'Hopital to show it:

$$\lim_{x\to\infty} \frac{x^3}{x^3+x} \overset{\text{LH}}{=} \lim_{x\to\infty} \frac{3x^2}{3x^2+1} \overset{\text{LH}}{=} \lim_{x\to\infty} \frac{6x}{6x} = 1.$$

Since the limit exists and is not 0 or ∞, we know our series and $\sum_{n=1}^{\infty} \frac{1}{n^3}$ either both converge or both diverge. Since the latter converges by the p-test, so does our series.

Example 16.5. *Use both comparison tests to determine whether* $\sum_{n=1}^{\infty} \frac{1}{n+\ln(n)}$ *converges or diverges.*

Solution. (1) **Direct Comparison**: As $n \to \infty$, the fraction behaves like $\frac{1}{n}$, and the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the p-test ($p = 1$). So we guess that it diverges and we want to find something which is smaller. The usual trick is to try and get rid of the term we don't care about (in this case, the $\ln(n)$), but in this case it doesn't work since removing it would make the denominator smaller, which is not what we want. Instead, we might try to replace $\ln(n)$ by n since that is the term we do care about, and we can since $\ln(n) \leq n$, so

$$\frac{1}{n+\ln(n)} \geq \frac{1}{n+n} = \frac{1}{2n}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges (p-test), our original series $\boxed{\text{diverges}}$.

(2) **Limit Comparison**: The first thing to do is to still look at behavior. It behaves like $1/n$, so let $a_n = \frac{1}{n+\ln(n)}$ (our original term) and let $b_n = \frac{1}{n}$ (the thing we want to compare it to). Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n+\ln(n)}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+\ln(n)}.$$

We could also use L'Hopital:

$$\lim_{x\to\infty} \frac{x}{x+\ln(x)} \overset{\text{LH}}{=} \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}} = 1,$$

since $1/x \to 0$. Since the limit exists and is not 0 or ∞, we know our series and $\sum_{n=1}^{\infty} \frac{1}{n}$ either both converge or both diverge. Since the latter diverges by the p-test, so does our series.

Remark. While it is true that $\ln(n) \leq n$ (or even $\ln(n) \leq \sqrt{n}$) for all $n \geq 1$, we actually have $\ln(n) \leq n^c$ for any positive exponent c, as long as n is big enough. Also, we have $n^a \leq b^n$ for any n large enough (here $a > 0$ and $b > 1$). So, for example, $n^{100} \leq e^n$ provided n is large enough, and $\ln(n) \leq n^{.001}$ provided n is large enough. The next example shows how useful these can be

Example 16.6. *Determine whether the series* $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^{3/2}}$ *converges or diverges.*

Solution. In terms of behavior, the $\ln(n)$ grows really slowly, so for all intents and purposes, this behaves like $1/n^{3/2}$, and the series $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges by the p-test with $p = 3/2$. Therefore, we guess that it converges, and we can use comparison to prove it. We want to find something bigger which converges. We can't just say $1/n^{3/2}$, since it isn't true that $\ln(n) \leq 1$ for all n. We also don't want to use $\ln(n) \leq n$, because this would mean we're comparing it to $\frac{n}{n^{3/2}} = \frac{1}{n^{1/2}}$, which doesn't converge (or rather, the corresponding series does not converge). So instead, we could say if n is large enough, $\ln(n) \leq n^{1/4}$. Since comparison test only requires us to have $a_n \leq b_n$ for all $n \geq M$ for some M, if M is the point at which $\ln(n) \leq n^{1/4}$ is true, then

$$\frac{\ln(n)}{n^{3/2}} \leq \frac{n^{1/4}}{n^{3/2}} = \frac{1}{n^{5/4}} \text{ for } n \geq M.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$ converges by the p-test, our original series $\boxed{\text{converges}}$.

And now we look at examples without being told what test to use.

Example 16.7. *Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n^n}$ converges or diverges.*

Solution. The sequence $\frac{1}{n^n}$ goes to 0 very, very quickly, so we should guess that it converges. We cannot integrate $\frac{1}{x^x}$, so the integral test is out. It isn't a p-series. We also might have a hard time finding a limit comparison here. So we are left with direct comparison. Luckily, if $n \geq 2$, then $\frac{1}{n^n} \leq \frac{1}{n^2}$, and since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-test, our series $\boxed{\text{converges}}$ by comparison.

Example 16.8. *Determine whether the series $\sum_{n=1}^{\infty} \frac{2n+1}{n^3-n+1}$ converges or diverges.*

Solution. This behaves likes $\frac{2n}{n^3} = \frac{2}{n^2}$, and so we should guess this converges by p-test. To prove it, we don't want to integrate that. We might try direct comparison, but it would be messy. But since we know it behave like $2/n^2$, we could just use limit comparison. Let $a_n = \frac{2n+1}{n^3-n+1}$ and $b_n = \frac{2}{n^2}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{2n+1}{n^3-n+1}}{\frac{2}{n^2}} = \lim_{n\to\infty} \frac{2n^3+n^2}{2n^3-2n+2}.$$

We know this limit should go to 1. Instead of using L'Hopital a few times to prove it, let's just multiply the top and bottom by $1/n^3$:

$$\lim_{n\to\infty} \frac{(2n^3+n^2)\frac{1}{n^3}}{(2n^3-2n+2)\frac{1}{n^3}} = \lim_{n\to\infty} \frac{2+\frac{1}{n}}{2-\frac{2}{n^2}+\frac{2}{n^3}} = 1,$$

since all the terms with n in the denominator now go to 0. By limit comparison, since $\sum_{n=1}^{\infty} \frac{2}{n^2}$ converges by the p-test, our series $\boxed{\text{converges}}$.

Example 16.9. *Determine whether the series $\sum_{n=1}^{\infty} \frac{n+1}{2n-3}$ converges or diverges.*

Solution. As $n \to \infty$, $\frac{n+1}{2n-3} \to \frac{1}{2} \neq 0$, and so the series $\boxed{\text{diverges}}$ by the divergence test (see previous section).

Example 16.10. *Determine whether the series $\sum_{n=1}^{\infty} \frac{2^n}{3^{n/2}}$ converges or diverges.*

Solution. The n's in the exponents everywhere make us think geometric series. Since $3^{n/2} = (\sqrt{3})^n$, the series is

$$\sum_{n=1}^{\infty} \frac{2^n}{3^{n/2}} = \sum_{n=1}^{\infty} \left(\frac{2}{\sqrt{3}}\right)^n.$$

Since $2/\sqrt{3} > 1$, this is a geometric series where the ratio is > 1, and so it $\boxed{\text{diverges}}$.

Example 16.11. *Determine whether the series $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$ converges or diverges.*

Solution. Any number of techniques would work here, but we notice that we can integrate $\frac{x}{(x^2+1)^2}$ with a simple u-sub, so let us use the integral test. We need to calculate

$$\int_1^\infty \frac{x}{(x^2+1)^2}dx = \lim_{M\to\infty}\int_1^M \frac{x}{(x^2+1)^2}dx.$$

Let $u = x^2+1$, $du = 2xdx$, so $xdx = \frac{1}{2}du$. Also changing the bounds yields

$$\int_1^M \frac{x}{(x^2+1)^2}dx = \frac{1}{2}\int_2^{M^2+1}\frac{du}{u^2} = -\frac{1}{2}\cdot\frac{1}{u}\Big|_2^{M^2+1} = -\frac{1}{2}\left(\frac{1}{M^2+1}-\frac{1}{2}\right).$$

As $M\to\infty$, the $\frac{1}{M^2+1}\to 0$, so the whole thing goes to $\frac{1}{4}$. The integral converges, and therefore the series $\boxed{\text{converges}}$.

Example 16.12. *Determine whether the series $\sum_{n=1}^\infty \frac{3+\sin(n)+(-1)^n}{n^2}$ converges or diverges.*

Solution. As $n\to\infty$, the $\sin(n)$ and $(-1)^n$ are inconsequential, as they are between -1 and 1. So this behaves like $3/n^2$, and that series converges by the p-test. So we guess that this converges. We cannot integrate that, and limit comparison would actually not work the way we would like (try it and see). So we can use direct comparison. We want to find something bigger which converges. Since $\sin(n)\le 1$ and $(-1)^n\le 1$, we have

$$\frac{3+\sin(n)+(-1)^n}{n^2}\le\frac{3+1+1}{n^2}=\frac{5}{n^2}.$$

Since $\sum_{n=1}^\infty \frac{5}{n^2}$ converges by the p-test, our series $\boxed{\text{converges}}$.

Example 16.13. *Determine whether the series $\sum_{n=1}^\infty \frac{2^n}{e^n-n}$ converges or diverges.*

Solution. As $n\to\infty$, the 2^n dominates on top and e^n on the bottom. Since $\sum_{n=1}^\infty \frac{2^n}{e^n} = \sum_{n=1}^\infty \left(\frac{2}{e}\right)^n$ converges (it is a geometric series and $2/e<1$), we guess that this thing converges. Again, we do not want to integrate that, and direct comparison is halted by the $-n$, since removing it would make the fraction smaller, which is not the way we want to go. So we try limit comparison instead, comparing it to $\frac{2^n}{e^n}$. So

$$\lim_{n\to\infty}\frac{\frac{2^n}{e^n-n}}{\frac{2^n}{e^n}} = \lim_{n\to\infty}\frac{e^n}{e^n-n}.$$

To show this is 1, we use L'Hopital:

$$\lim_{x\to\infty}\frac{e^x}{e^x-x}\overset{\text{LH}}{=}\lim_{x\to\infty}\frac{e^x}{e^x-1}\overset{\text{LH}}{=}\lim_{x\to\infty}\frac{e^x}{e^x}=1.$$

Since $\sum_{n=1}^\infty \frac{2^n}{e^n}$ converges (geometric series with $r<1$), our series $\boxed{\text{converges}}$ by limit comparison.

Example 16.14. *Determine whether the series $\sum_{n=1}^\infty \cos(1/n)$ converges or diverges.*

Solution. As $n\to\infty$, $1/n\to 0$, so $\cos(1/n)\to\cos(0)=1\ne 0$. So the series $\boxed{\text{diverges}}$ by the divergence test.

17 Alternating Series (11.4)

Summary 17.1. *(1) A series $\sum_{n=1}^\infty a_n$* ***converges absolutely*** *if $\sum_{n=1}^\infty |a_n|$ converges.*

(2) Theorem: If $\sum_{n=1}^\infty |a_n|$ converges, then $\sum_{n=1}^\infty a_n$ converges.

(3) A series $\sum_{n=1}^\infty a_n$ ***converges conditionally*** *if $\sum_{n=1}^\infty a_n$ converges but $\sum_{n=1}^\infty |a_n|$ diverges.*

(4) Alternating Series Test (a.k.a. Leibniz Test): Suppose $\{a_n\}$ is a sequence of positive terms which is decreasing and converging to 0. Then $\sum_{n=1}^\infty (-1)^{n-1}a_n$ converges.

(5) *Under the conditions of item (4), if $S = \sum_{n=1}^{\infty}(-1)^{n-1}a_n$, then the error from the N-th partial sum satisfies $|S - S_N| < a_{N+1}$.*

Remark. (1) It is important to note that $\sum_{n=1}^{\infty}|a_n|$ diverging tells you nothing about $\sum_{n=1}^{\infty}a_n$.

(2) It is also crucial that, in item (4), the $a_n \to 0$. If they don't tend to 0, the series automatically diverges by the divergence test from 11.2.

(3) It doesn't much matter whether it is $(-1)^{n-1}$ or $(-1)^n$ in item (4).

First, a simple example:

Example 17.2. *Consider the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$. Show that the series converges conditionally, but not absolutely. Then find N so that $|S - S_N| < 10^{-3}$, where S is the sum of the series.*

Solution. To check absolute convergence convergence, we need to see if $\sum_{n=1}^{\infty}\left|\frac{(-1)^{n-1}}{n}\right|$ converges or not. Note

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n-1}}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n},$$

and this series diverges by the p-test. Therefore the series does not converge absolutely.
We can use the Alternating Series Test to prove it converges conditionally. Here, $a_n = \frac{1}{n}$ (you always let a_n be the positive stuff without the $(-1)^n$). Note that $a_n > 0$ for all n, and that $a_n \to 0$. We also need $\{a_n\}$ to be decreasing, but $a_{n+1} < a_n$ since $\frac{1}{n+1} < \frac{1}{n}$. Therefore the conditions of ther theorem are satisfied, and so $\sum_{n=1}^{\infty}(-1)^{n-1}a_n = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ converges conditionally.
For the last part of the question, item (5) in the summary says that $|S - S_N| < a_{N+1}$. In this example, that means $|S - S_N| < \frac{1}{N+1}$. Since we want $|S - S_N| < 10^{-3}$, we can just set $\frac{1}{N+1} < 10^{-3}$, and solve for N. Solving, we get

$$\frac{1}{N+1} < 10^{-3} \implies N + 1 > 1000 \implies N > 999.$$

So, for example $N = 1000$ works. What this says is that the 1000-th partial sum is within 10^{-3} of the actual value.

Example 17.3. *Determine whether the series $\sum_{n=1}^{\infty}\frac{(-1)^n n}{n+1}$ converges absolutely, conditionally, or not at all.*

Solution. Note that, as $n \to \infty$, $\frac{n}{n+1} \to 1$, since

$$\lim_{x\to\infty}\frac{x}{x+1} \overset{\text{LH}}{=} \lim_{x\to\infty}\frac{1}{1} = 1$$

by L'Hopital's rule. In particular, the general term $\frac{(-1)^n n}{n+1} \not\to 0$, so the series diverges by the divergence test.

Example 17.4. *Determine whether the series $\sum_{n=1}^{\infty}\frac{(-3)^n}{\pi^n}$ converges absolutely, conditionally, or not at all.*

Solution. In this case, the general term does go to 0 since $\pi > 3$, so we at least have the chance of convergence. We first check absolute convergence. We need to check whether

$$\sum_{n=1}^{\infty}\left|\frac{(-3)^n}{\pi^n}\right| = \sum_{n=1}^{\infty}\frac{3^n}{\pi^n}$$

converges or not. But $\sum_{n=1}^{\infty}\frac{3^n}{\pi^n}$ is a geometric series with ratio $3/\pi < 1$, so it will converge. Therefore, our original series $\boxed{\text{converges absolutely}}$.

Example 17.5. *Determine whether the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n^2+1}}$ converges absolutely, conditionally, or not at all.*

Solution. Again, the general term goes to 0, so we cannot use the divergence test. To check absolute convergence, note
$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n^2+1}} \right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}.$$
This behaves like $1/n$, so it should diverge. To show it, we could use either direct or limit comparison:

(1) Direct: We can't just remove the 1 from the denominator, since this would make our fraction bigger. Instead, we try and get the 1 in terms of the dominant term. Since $1 \le n^2$ (as $n \ge 1$ in the series), we can say
$$\frac{1}{\sqrt{n^2+1}} \ge \frac{1}{\sqrt{n^2+n^2}} = \frac{1}{\sqrt{2n^2}} = \frac{1}{\sqrt{2}} \cdot \frac{1}{n}.$$
The inequality is justified by the fact that we made the denominator bigger by replacing the 1 by the n^2. Since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{2}} \cdot \frac{1}{n}$ diverges by the p-test ($p = 1$), our series diverges by comparison.

(2) Limit comparison: Compare to $\frac{1}{n}$. Note
$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^2+1}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}}.$$
To evaluate this, we can multiply the top and bottom by $1/n$:
$$\lim_{n\to\infty} \frac{n \cdot \frac{1}{n}}{\sqrt{n^2+1} \cdot \frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{n^2+1} \cdot \sqrt{\frac{1}{n^2}}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}} = 1.$$
Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so does our series.

All this shows is that it does not converge absolutely. To check conditional convergence, we can use the Leibniz test. Let $a_n = \frac{1}{n^2+1}$ (again, the stuff without the $(-1)^n$). Then $a_n > 0$. To see $a_n \to 0$, note that by using the same manipulation as in the Limit Comparison test above,
$$\lim_{n\to\infty} \frac{1 \cdot \frac{1}{n}}{\sqrt{n^2+1} \cdot \frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\sqrt{1+\frac{1}{n^2}}} = \frac{0}{1} = 0.$$
Finally, we need $a_{n+1} < a_n$. But this is true, since
$$(n+1)^2 > n^2 \implies (n+1)^2 + 1 > n^2 + 1 \implies \sqrt{(n+1)^2+1} > \sqrt{n^2+1} \implies \frac{1}{\sqrt{(n+1)^2+1}} < \frac{1}{\sqrt{n^2+1}}.$$
By the Leibniz test, the series $\boxed{\text{converges conditionally}}$.

18 Ratio and Root Test (11.5)

The last two tests are the ratio and root test:

Summary 18.1. *(1) Ratio Test: Let $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ (provided the limit exists). Then*

- *(i) If $\rho < 1$, then $\sum a_n$ converges absolutely.*
- *(ii) If $\rho > 1$, then $\sum a_n$ diverges.*
- *(iii) If $\rho = 1$, test is inconclusive.*

(2) Root Test: Let $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ (provided the limit exists). Then:

- *(i) If $L < 1$, then $\sum a_n$ converges absolutely.*

(ii) *If $L > 1$, then $\sum a_n$ diverges.*

(iii) *if $L = 1$, the test is inconclusive.*

Remark. For ratio test, the biggest clues are factorials and exponentials. For root test, you are looking for n's in the exponents.

Example 18.2. *Determine whether the series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges or diverges.*

Solution. Here we can use the ratio test. If $a_n = \frac{n^2}{2^n}$, then

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}} = \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} = \frac{(n+1)^2}{2n^2}.$$

As $n \to \infty$, you can use L'Hopital's rule to show the limit is $1/2$. So $\rho = 1/2 < 1$, meaning the series $\boxed{\text{converges}}$ by the ratio test.

Example 18.3. *Determine whether the series $\sum_{n=1}^{\infty} \frac{(-2)^n}{(2n)!}$ converges absolutely, conditionally, or neither.*

Solution. The factorials and exponentials tell us to try ratio test. Again,

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(-2)^{n+1}}{(2(n+1))!}}{\frac{(-2)^n}{(2n)!}} = \frac{(-2)^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{(-2)^n} = \frac{2}{(2n+1)(2n+2)},$$

since the absolute value removed the $-$ sign and $\frac{(2n)!}{(2n+2)!} = \frac{1}{(2n+2)(2n+1)}$. Now, as $n \to \infty$, this goes to 0, since multiplying the top and bottom by $1/n^2$ gives:

$$\frac{2}{4n^2 + 6n + 2} = \frac{\frac{2}{n^2}}{4 + \frac{6}{n} + \frac{2}{n^2}} \to \frac{0}{4} = 0.$$

Therefore $\rho = 0 < 1$, so the series converges absolutely.

Example 18.4. *Determine whether $\sum_{n=1}^{\infty} (-1)^n \left(\frac{2n}{n+1}\right)^n$ converges absolutely, conditionally, or not at all.*

Solution. The n in the exponent tells us to try root test. Since $|a_n| = \left(\frac{2n}{n+1}\right)^n$, taking n-th roots gives $\sqrt[n]{a_n} = \frac{2n}{n+1}$. As $n \to \infty$, L'Hopital gives this limit as 2, so $L = 2 > 1$ means the series $\boxed{\text{diverges}}$.

Example 18.5. *Show for any r, the series $\sum_{n=1}^{\infty} \frac{r^n}{n^n}$ converges absolutely.*

Solution. Here, $|a_n| = \frac{|r|^n}{n^n}$, so $\sqrt[n]{a_n} = \frac{|r|}{n}$. As $n \to \infty$, this goes to 0, since $|r|$ is some number. Therefore $L = 0 < 1$, so the series will converge absolutely.

Remark. This finishes off the tests for convergence. To summarize we have:

(1) Divergence Test: Make sure $a_n \to 0$.

(2) Positive terms:

  (a) p-Test: This you can only use for $1/n^p$, so should be very recognizable.

  (b) Integral Test: If you feel like you can integrate the function, use this test. Make sure the function is positive (no $(-1)^n$'s floating around).

  (c) Comparison Test: This doesn't really have a distinguished look, but is really the only one which doesn't, so you can get to this by process of elimination.

  (d) Limit Comparison: Use this if you can determine behavior easily (e.g. $\frac{n^2+n+1}{n^3+1}$). It won't always help, but you can try it.

(3) Leibniz Test: This must be an alternating series, and you should only do this after you check for absolute convergence.

(4) Ratio Test: Exponentials and factorials are big clues here (as in the examples above).

(5) Root Test: When you have n's in the exponents, you can try this test.

It is really important you practice determining convegence of random series, since practice is the only way to make you more comfortable identifying which test to use.

19 Taylor Polynomials (9.4)

Now you will see why all your work for infinite series was worth it: Taylor Series. First, we need Taylor polynomials.

Summary 19.1. *Let $f(x)$ be some function.*

(1) The n-th Taylor polynomial centered at $x = a$ is the polynomial

$$T_n(x) = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k.$$

(2) If $a = 0$, $T_n(x)$ is called a Maclaurin polynomial.

(3) If $f^{(n+1)}(x)$ exists and is continuous, then the error for $T_n(x)$ is

$$|T_n(x) - f(x)| \leq K\frac{|x-a|^{n+1}}{(n+1)!},$$

where K is a number with $|f^{(n+1)}(u)| \leq K$ for all u between a and x.

Remark. What you're doing with Taylor polynomials is approximating some (presumably complicated) function by polynomials, which are easier to understand.

Example 19.2. *If $f(x) = e^x$, find:*

(a) $T_3(x)$ centered at $a = 0$

(b) The maximum possible error for $|T_3(.1) - f(.1)|$

(c) $T_n(x)$ centered at $a = 0$

Solution. (a) To find the third Maclaurin polynomial, we need three derivatives. In this case, $f(x) = f'(x) = f''(x) = f'''(x) = e^x$. So in each case, when we plug in $x = 0$, we get $f(0) = f'(0) = f''(0) = f'''(0) = 1$. So using the formula from (1) in the summary with $n = 3$, we get

$$T_3(x) = 1 + 1(x-0) + \frac{1}{2!}(x-0)^2 + \frac{1}{3!}(x-0)^3 = \boxed{1 + x + \frac{x^2}{2} + \frac{x^3}{6}}.$$

(b) We are using the error bound. Here, $a = 0$, and x will always be the value you're trying to approximate. Since we're using $T_3(x)$ to approximate $f(.1)$, $x = .1$ in this scenario. Finally, we need K. By definition K is a number with $|f^{(4)}(u)| \leq K$ between a and x, so in this case on $[0, .1]$. This is similar to finding K for trapezoidal rule, except you use a different derivative. In this case, $|f^{(4)}(x)| = |e^x| = e^x$, and on $[0, .1]$, the largest this can be is at $x = .1$ since e^x is increasing. So

$$|f^{(4)}(x)| \leq e^{.1} \leq e^1 \leq 3,$$

so we can use $K = 3$. (Notice we could have used $K = e^{.1}$ if we wished, this was just done to illustrate that it doesn't matter whether we choose the smallest value of K or not.) So using the error bound with $K = 3$, $x = .1$, $a = 0$, and $n = 3$ in the formula from (2) in the summary above, we get

$$\boxed{|T_3(.1) - f(.1)| \leq 3\frac{(.1-0)^4}{4!}}.$$

(c) To find $T_n(x)$, we need to somehow find a formula for $f^{(n)}(a)$ in terms of n. This will probably be the hardest thing from this section, only because it involves some pattern recognition. In this case, it is easy, since all derivatives are e^x, meaning $f^{(n)}(0) = e^0 = 1$. Therefore,

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = \boxed{\sum_{k=0}^{n} \frac{1}{k!}x^k}.$$

Example 19.3. *Consider $f(x) = \frac{1}{(x+1)^2}$. Find:*

(a) $T_2(x)$ centered at $a = 1$, and use it to approximate $\frac{1}{2.1^2}$

(b) Find the maximum possible error from the approximation in (a)

(c) Find $T_n(x)$ for all n.

Solution. (a) To find $T_2(x)$, we take two derivatives:

$$f'(x) = -\frac{2}{(x+1)^3}, \quad f''(x) = \frac{2 \cdot 3}{(x+1)^4},$$

so $f(1) = \frac{1}{4}$, $f'(1) = -\frac{2}{8} = -\frac{1}{4}$, and $f''(1) = \frac{6}{16} = \frac{3}{8}$. Therefore

$$T_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 = \frac{1}{4} - \frac{1}{4}(x-1) + \frac{3/8}{2}(x-1)^2 = \boxed{\frac{1}{4} - \frac{1}{4}(x-1) + \frac{3}{16}(x-1)^2}.$$

To approximate $\frac{1}{2.1^2}$, notice that this is $f(1.1)$. Therefore, we plug in $x = 1.1$ into $T_2(x)$:

$$\boxed{f(1.1) \approx T_2(1.1) = \frac{1}{4} - \frac{1}{4}(1.1-1) + \frac{3}{16}(1.1-1)^2}.$$

(b) To approximate the error, we use the error bound. We need to find K such that $|f'''(x)| \leq K$ for all $x \in [1, 1.1]$ (i.e. for all x in the interval between a and the value you're approximating). In this case,

$$|f'''(x)| = \left| -\frac{2 \cdot 3 \cdot 4}{(x+1)^5} \right| = \frac{24}{(x+1)^5}.$$

This is a decreasing function, so it will have its maximum at the left endpoint, i.e. when $x = 1$, so K can be taken to be

$$K = \frac{24}{(1+1)^5} = \frac{24}{32} = \frac{3}{4}.$$

(Again, we could take any higher value as well.) Using the error bound with $K = \frac{3}{4}$, $n = 2$, $a = 1$, and $x = 1.1$ gives

$$\boxed{|T_2(1.1) - f(1.1)| \leq \frac{3}{4} \cdot \frac{(1.1-1)^3}{3!}}.$$

(c) To find a formula for $T_n(x)$, we need a general term for $f^{(k)}(a) = f^{(k)}(1)$. We have two derivatives above. Take a few more and see that

$$f(1) = \frac{1}{2^2}, \quad f'(1) = -\frac{2}{2^3}, \quad f''(1) = \frac{2\cdot 3}{2^4}, \quad f'''(1) = -\frac{2\cdot 3\cdot 4}{2^5}, \quad f^{(4)}(1) = \frac{2\cdot 3\cdot 4\cdot 5}{2^6}, \ldots.$$

Identify the changing parts. We see an alternating $\pm$, so we will get $(-1)^k$ or $(-1)^{k+1}$. If $k = 0$, the term is positive, so we take $(-1)^k$. In the denominator, we have a power of 2. For the 0-th derivative, the power is 2. In the first derivative, the power is 3, so in each case, the power is one more than the derivative we're taking. So we get 2^{k+2} in the denominator. On top, we see factorials. In the second derivative, we have 3!. In the third derivative, it is 4!, so in general, it will be $(k+1)!$. So, $f^{(k)}(1) = \frac{(-1)^k (k+1)!}{2^{k+2}}$. Using the general formula for the Taylor formula, we get,

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(1)}{k!}(x-1)^k = \sum_{k=0}^{n} \frac{(-1)^k (k+1)!}{2^{k+2}} \cdot \frac{1}{k!}(x-1)^k = \boxed{\sum_{k=0}^{n} \frac{(-1)^k (k+1)}{2^{k+2}}(x-1)^k}.$$

Example 19.4. *If $f(x) = \sin(2x)$, find $T_3(x)$ centered at $a = \pi/4$, and then find $T_n(x)$ for any n.*

Solution. (a) To find $T_3(x)$, first take three derivatives of $f(x)$:

$$f'(x) = 2\cos(2x), \quad f''(x) = -2^2\sin(2x), \quad f'''(x) = -2^3\cos(2x).$$

So at $a = \pi/4$, we have

$$f(\pi/4) = \sin(\pi/2) = 1, \quad f'(\pi/4) = 0, \quad f''(\pi/4) = -4, \quad f'''(\pi/4) = 0,$$

since $\cos(\pi/2) = 0$. So

$$T_3(x) = f(\pi/4) + f'(\pi/4)(x-\pi/4) + \frac{f''(\pi/4)}{2!}(x-\pi/4)^2 + \frac{f'''(\pi/4)}{3!}(x-\pi/4)^3 = 1 - \frac{4}{2!}(x-\pi/4)^2 = \boxed{1 - 2(x-\pi/4)^2}.$$

(b) To find $T_n(x)$, we need a formula for $f^{(n)}(\pi/4)$. We have three derivatives above, and already we notice that every other derivative will be 0. We can calculate a couple more derivatives:

$$f^{(4)}(x) = 2^4\sin(2x), \quad f^{(5)}(x) = 2^5\cos(2x), \quad f^{(6)}(x) = -2^6\sin(2x),$$

and so on. So at $a = \pi/4$, the sequence of derivatives we have is:

$$1, 0, -2^2, 0, 2^4, 0, -2^6, 0, \ldots.$$

Now, you have to deal with the fact that we only have nonzero terms in the even slots (namely, $k = 0, 2, 4, 6, \ldots$). Even numbers have the form $2k$, for k an integer, and odd numbers have the form $2k+1$, where k is some integer. The only nonzero derivatives come from $f^{(2k)}(\pi/4)$, so let's focus on the even terms:

$$1, -2^2, 2^4, -2^6, \ldots.$$

We see the alternating $\pm$, so we're going to get a $(-1)^k$ or $(-1)^{k+1}$. If $k = 0$, we have a positive term, so that tells us to take the $(-1)^k$. We then see we have 2 to an appropriate power. In the 0-th derivative, the power is 0. In the second derivative, the power is 2, in the fourth derivative the power is 4, so in the $2k$-th derivative, the power will just be $2k$. So

$$f^{(2k)}(\pi/4) = (-1)^k \cdot 2^{2k}.$$

Again, all the odd derivatives are 0 at $\pi/4$. So instead of giving the polynomial $T_n(x)$, we can give the polynomials $T_{2n}(x)$ and $T_{2n+1}(x)$. The formula for $T_{2n}(x)$ is

$$T_{2n}(x) = f(\pi/4) + f'(\pi/4)(x-\pi/4) + \frac{f''(\pi/4)}{2!}(x-\pi/4)^2 + \ldots + \frac{f^{(2n)}(\pi/4)}{(2n)!}(x-\pi/4)^{2n}.$$

All the odd terms are 0, and we have a formula for $f^{(2n)}(\pi/4)$, namely $f^{(2n)}(\pi/4) = (-1)^n \cdot 2^{2n}$. So

$$T_{2n}(x) = 1 - 2(x - \pi/4)^2 + \ldots + \frac{(-1)^n \cdot 2^{2n}}{(2n)!}(x - \pi/4)^{2n}.$$

This only gives the even order Taylor polynomials. We should also give the odd order ones. But the next Taylor expansion, namely $T_{2n+1}(x)$, will be the same as $T_{2n}(x)$ since $T_{2n+1}(x) = T_{2n}(x) + \frac{f^{(2n+1)}(\pi/4)}{(2n+1)!}(x - \pi/4)^{2n+1}$ and $f^{(2n+1)}(\pi/4) = 0$, so

$$T_{2n}(x) = T_{2n+1}(x).$$

So this describes every Taylor polynomial.

$$\boxed{T_{2n}(x) = T_{2n+1}(x) = 1 - 2(x - \pi/4)^2 + \ldots + \frac{(-1)^n \cdot 2^{2n}}{(2n)!}(x - \pi/4)^{2n}}.$$

Example 19.5. *Consider* $f(x) = \ln(1 - x)$ *with* $a = 0$. *Find* $T_2(x)$, $T_n(x)$, *and then find* n *so that* $|\ln(.8) - T_n(.2)| < 10^{-3}$.

Solution. (a) To calculate $T_2(x)$, we take two derivatives of $f(x)$:

$$f'(x) = -\frac{1}{1-x}, \quad f''(x) = -\frac{1}{(1-x)^2}.$$

Here, $a = 0$, so we observe

$$f(0) = \ln(1) = 0, \quad f'(0) = -1, \quad f''(0) = -1.$$

Therefore,

$$T_2(x) = f(0) + f'(0)(x - 0) + \frac{f''(0)}{2!}(x - 0)^2 = \boxed{-x - \frac{x^2}{2}}.$$

(b) To find $T_n(x)$, we need a general formula for the k-th derivative of $f(x)$, $f^{(k)}(x)$. We've taken two derivatives, let's take a few more to see the pattern,

$$f'(x) = -\frac{1}{1-x}, \quad f''(x) = -\frac{1}{(1-x)^2}, \quad f'''(x) = -\frac{1 \cdot 2}{(1-x)^3}, \quad f^{(4)}(x) = -\frac{1 \cdot 2 \cdot 3}{(1-x)^4}.$$

We begin to see a pattern for the derivatives. We always have a minus sign. In the bottom, the power of $(1 - x)$ is going up by 1 each time. For the first derivative, the power is 1, for the second, the power is 2, for the third derivative it is 3. So for the k-th derivative, the power of $(1 - x)$ will be k. For the numerator, we see a factorial pattern emerging. In the third derivative we have 2!, in the fourth derivative, it is 3!, so for the k-th derivative, it will be $(k - 1)!$. But notice this only works for the derivatives, since the function itself does not follow this pattern (i.e. $\ln(1 - x)$ does not follow this pattern). So we can only say

$$f^{(k)}(x) = -\frac{(k-1)!}{(1-x)^k}, \quad k \geq 1.$$

So at 0,

$$f^{(k)}(0) = -\frac{(k-1)!}{(1-0)^k} = -(k-1)!, \quad k \geq 1.$$

We know $T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k$, but we only have a formula for $f^{(k)}(a)$ if $k \geq 1$. So we can write

$$T_n(x) = f(0) + \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k,$$

where we just isolate the $k = 0$ term and then start the sum from $k = 1$. (Note: We're doing this only because our k=0 term did not follow the pattern.) Since $f(0) = 0$, we have

$$T_n(x) = 0 + \sum_{k=1}^{n} \frac{-(k-1)!}{k!} x^k = -\sum_{k=1}^{n} \frac{1}{k} x^k = \boxed{-x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots - \frac{x^n}{n}}.$$

(c) So this is an error bound problem, observe that $\ln(.8) = f(.2)$, so we're trying to find n so that $|f(.2) - T_n(.2)| < 10^{-3}$. Remember that the error bound for this would be

$$|f(.2) - T_n(.2)| \leq K \frac{(.2-0)^{n+1}}{(n+1)!},$$

where $|f^{(n+1)}(x)| \leq K$ for $x \in [0, .2]$. We already found a formula for the k-th derivative, namely $f^{(k)}(x) = -\frac{(k-1)!}{(1-x)^k}$, so

$$f^{(n+1)}(x) = -\frac{(n+1-1)!}{(1-x)^{n+1}} = -\frac{n!}{(1-x)^{n+1}}$$

(just plug in $k = n+1$ everywhere in the formula). So

$$|f^{(n+1)}(x)| = \frac{n!}{(1-x)^{n+1}}.$$

This is going to be biggest when $1 - x$ is smallest, so at $x = 0$. So

$$\frac{n!}{(1-x)^{n+1}} \leq \frac{n!}{(1-0)^{n+1}} = n!.$$

So we can let $K = n!$, and then the error bound becomes

$$K \frac{(.2-0)^{n+1}}{(n+1)!} = n! \frac{.2^{n+1}}{(n+1)!} = \frac{.2^{n+1}}{n+1} = \frac{1}{(n+1)5^{n+1}}$$

as $.2 = \frac{1}{5}$. So we want to find n so that

$$\frac{1}{(n+1)5^{n+1}} < 10^{-3} = \frac{1}{1000}.$$

We can't solve for n explicitly, but we can plug in some numbers until we find it: If $n = 2$, then $\frac{1}{(n+1)5^{n+1}} = \frac{1}{3 \cdot 5^3} = \frac{1}{375}$, so this is no good. If $n = 3$, then $\frac{1}{(n+1)5^{n+1}} = \frac{1}{4 \cdot 5^4} < \frac{1}{1000}$, so $\boxed{n = 4}$ is the one we want.

There are some Taylor polynomials which are just good to know:

(1) $e^x, a = 0 \to T_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$

(2) $\sin(x), a = 0 \to T_{2n+1}(x) = T_{2n+2}(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$

(3) $\cos(x), a = 0 \to T_{2n}(x) = T_{2n+1}(x) = \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!}$

20 Power Series (11.6)

Definition 20.1. *A power series about c is a series of the form $\sum_{n=0}^{\infty} a_n (x-c)^n$, where the a_n are numbers.*

Example 20.2. *The infinite series $\sum_{n=0}^{\infty} x^n$ is a power series about 0, with the $a_n = 1$ for all n.*

Example 20.3. *The sum $\sum_{n=0}^{\infty} \frac{n!}{3^n} (x-5)^{2n}$ is a power series about 5 (even though the exponent is $2n$ instead of n.)*

A general fact is the following:

Proposition 20.4. *Let $F(x)$ be a power series about c. Then there exists $0 \le R$ (called the* ***radius of convergence****; may be infinite) so that $F(x)$ converges absolutely for $|x-c| < R$ and diverges for $|x-c| > R$.*

You usually get R using the ratio test:

Example 20.5. *Determine the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$.*

Solution. We will use the ratio test with $a_n = \frac{x^n}{n}$. We compute

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{x^{n+1}}{2^{n+1}}}{\frac{x^n}{2^n}} = \lim_{n\to\infty} \frac{x^{n+1}}{2^{n+1}} \cdot \frac{2^n}{x^n} = \lim_{n\to\infty} \frac{x}{2} = \frac{x}{2} .$$

Now, ratio test says that we will converge absolutely if $\rho < 1$, and since $\rho = \frac{|x|}{2}$, we need $\frac{|x|}{2} < 1$, or $|x| < 2$. Since this is the same as $|x - 0| < 2$ and our center is 0, the radius of convergence is $\boxed{R = 2}$.

Now, there is a slight difference between radius of convergence and **interval of convergence**. We found, for example, that the radius of convergence for the previous example is 2. But the interval of convergence is the actual interval of x values for which the series converges.

Example 20.6. *Determine the interval of convergence for the series in the previous problem, namely $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$.*

Solution. So you first find the radius of convergence, which we said was 2. So we know we will converge if $|x| < 2$, i.e. if $-2 < x < 2$. Moreover, if $|x| > 2$, then $\rho > 1$ in the previous problem, and radius test tells us we will diverge. So the only question at the moment is whether the series converges at the endpoints, namely ± 2 (this would correspond to $\rho = 1$ in the ratio test, which we know is inconclusive). So check convergence at the endpoints, we plug in $x = 2$ and $x = -2$ into the series, and determine whether this infinite series converges or diverges. For example, if $x = 2$, then the power series is

$$\sum_{n=0}^{\infty} \frac{2^n}{2^n} = \sum_{n=0}^{\infty} 1,$$

which diverges by the divergence test (as the general term, 1 is not going to 0). So we do not converge at $x = 2$. Similarly, at $x = -2$, the power series becomes

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{2^n} = \sum_{n=0}^{\infty} (-1)^n,$$

which also diverges by the divergence test. Therefore the series does not converge at either endpoint, and so the interval of convergence is precisely $\boxed{-2 < x < 2}$.

So there are two steps to find the interval: find the radius of convergence, and check the endpoints.

Example 20.7. *Determine the radius and interval of convergence for the series $\sum_{n=1}^{\infty} \frac{8^n}{n}(x-2)^n$*

Solution. First, identify the center. In this case, it is 2. We use the ratio test with $a_n = \frac{8^n}{n}(x-2)^n$ to get

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \\
&= \lim_{n\to\infty} \frac{\frac{8^{n+1}(x-2)^{n+1}}{n+1}}{\frac{8^n(x-2)^n}{n}} \\
&= \lim_{n\to\infty} \frac{8^{n+1}(x-2)^{n+1}}{n+1} \cdot \frac{n}{8^n(x-2)^n} \\
&= \lim_{n\to\infty} 8(x-2)\frac{n}{n+1} \\
&= |8(x-2)| \lim_{n\to\infty} \frac{n}{n+1} \\
&= 8|x-2|,
\end{aligned}$$

since $\lim_{n\to\infty} \frac{n}{n+1} = 1$. So for convergence, we need $\rho = 8|x-2| < 1$, so $|x-2| < \frac{1}{8}$. Therefore the radius of convergence is $\boxed{R = \frac{1}{8}}$. To find the interval, right now we have convergence for $|x-2| < 1/8$, which means

$$-\frac{1}{8} < x - 2 < \frac{1}{8} \implies \frac{15}{8} < x < \frac{17}{8},$$

where this last inequality was gotten by adding 2 to every side in the first inequality. So the only question is whether we have convergence at $x = 15/8$ and $x = 17/8$. Plugging the latter point into the series gives

$$\sum_{n=1}^{\infty} \frac{8^n}{n} \left(\frac{17}{8} - 2\right)^n = \sum_{n=1}^{\infty} \frac{8^n}{n} \left(\frac{1}{8}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n}.$$

But this series diverges by the p-test, so we do not converge at $x = 17/8$. At $x = 15/8$, we have

$$v \sum_{n=1}^{\infty} \frac{8^n}{n} \left(\frac{15}{8} - 2\right)^n = \sum_{n=1}^{\infty} \frac{8^n}{n} \left(-\frac{1}{8}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

and this series does converge by the Leibniz test (you should show this). Therefore we do have convergence at $x = 15/8$, and so the interval of convergence is $\boxed{\frac{15}{8} \leq x < \frac{17}{8}}$.

Example 20.8. *Determine the radius and interval of convergence for the series* $\sum_{n=1}^{\infty} n!(2x+1)^n$.

Solution. Again, first identify the center. In this case, the center is $-1/2$ since this is what makes the $2x+1$ equal to 0. So we want R so that the series converges for $|x - \frac{1}{2}| < R$. Again, using the ratio test with $a_n = n!(2x+1)^n$ gives

$$\begin{aligned}
\rho &= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \\
&= \lim_{n\to\infty} \frac{(n+1)!(2x+1)^{n+1}}{n!(2x+1)^n} \\
&= \lim_{n\to\infty} |(n+1)(2x+1)| \\
&= |2x+1| \lim_{n\to\infty} n+1.
\end{aligned}$$

Now, as long as $x \neq -1/2$ (i.e. $2x+1 \neq 0$), that limit will be infinite, so if $x \neq -1/2$, the series is guaranteed to diverge (since ρ cannot be less than 1). We do, however, have convergence at $x = -1/2$ (since we always have convergence at the center). So the interval of convergence is just $x = -1/2$, and the radius of convergence is 0.

Remark. If, when doing ratio test, you get a limit of 0 regardless of x, then the radius of convergence would be infinite and the series would converge for all x.

Example 20.9. *Determine the radius and interval of convergence for the series* $\sum_{n=0}^{\infty} \frac{x^{2n}}{(-3)^n}$

Solution. Here, the center is $a = 0$. Using ratio test with $a_n = \frac{x^{2n}}{(-3)^n}$ gives

$$
\begin{aligned}
\rho &= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \\
&= \lim_{n\to\infty} \frac{\frac{x^{2(n+1)}}{(-3)^{n+1}}}{\frac{x^{2n}}{(-3)^n}} \\
&= \lim_{n\to\infty} \frac{x^{2n+2}}{(-3)^{n+1}} \cdot \frac{(-3)^n}{x^{2n}} \\
&= \lim_{n\to\infty} \frac{x^2}{-3} \\
&= \frac{x^2}{3}.
\end{aligned}
$$

This will converge if $\rho < 1$, so if $\frac{x^2}{3} < 1$, or $x^2 < 3$. It is important that you realize the radius is not 3 here, since the radius is given by an inequality of the form $|x| < R$ (since the center here is 0). So we notice $x^2 < 3 \implies |x| < \sqrt{3}$ (since $\sqrt{x^2} = |x|$). So $\boxed{R = \sqrt{3}}$. The interval we have so far is $|x| < \sqrt{3}$, or $-\sqrt{3} < x < \sqrt{3}$. We need to check convergence at $x = \pm\sqrt{3}$. At $x = \sqrt{3}$, the series is

$$\sum_{n=0}^{\infty} \frac{\sqrt{3}^{2n}}{(-3)^n} = \sum_{n=0}^{\infty} \frac{3^n}{(-3)^n} = \sum_{n=0}^{\infty} (-1)^n,$$

which diverges by divergence test. So we do not have convergence at $x = \sqrt{3}$. At the other endpoint, we get

$$\sum_{n=0}^{\infty} \frac{(-\sqrt{3})^{2n}}{(-3)^n} = \sum_{n=0}^{\infty} \frac{3^n}{(-3)^n} = \sum_{n=0}^{\infty} (-1)^n,$$

which again diverges. So we do not have convergence at either endpoint, and so the interval of convergence is $\boxed{-\sqrt{3} < x < \sqrt{3}}$.

Remark. In general, when checking the endpoints, you will get an infinite series which requires a convergence test from sections 11.2-11.5, so be prepared to use those.

We can use the formula for a sum of a geometric series to allow us to write some functions as power series. Consider the function $f(x) = \frac{1}{1-x}$. This looks like the sum of a geometric series $\sum_{n=0}^{\infty} cr^n = \frac{c}{1-r}$, where $c = 1$ and $r = x$. So we can treat $\frac{1}{1-x}$ as the sum of the geometric series, and write

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots + x^n + \ldots = \sum_{n=0}^{\infty} x^n.$$

This will be convergent when $|r| = |x| < 1$. This will allow us to find the power series of other functions.

Example 20.10. *Find a power series centered at* 0 *for* $f(x) = \frac{1}{1-3x}$. *What is the radius of convergence?*

Solution. Again, we can treat this as a geometric series with first term 1 and ratio $r = 3x$. So

$$\frac{1}{1-3x} = 1 + 3x + (3x)^2 + \ldots + (3x)^n + \ldots = 1 + 3x + 9x^2 + \ldots + 3^n x^n + \ldots = \sum_{n=0}^{\infty} 3^n x^n.$$

This will be convergent if $|r| = |3x| < 1$, meaning $|x| < 1/3$, so the radius of convergence is $1/3$.

Example 20.11. *Find a power series for* $f(x) = \frac{1}{3+x^3}$ *centered at* 0. *What is the radius of convergence?*

Solution. Here, we can't quite use the geometric series formula because of the 3 in the denominator. If it were a 1, then we could, so we can factor out a 3 from the denominator:

$$f(x) = \frac{1}{3+x^3} = \frac{1}{3\left(1+\frac{x^3}{3}\right)} = \frac{1}{3} \cdot \frac{1}{1+\frac{x^3}{3}}.$$

Now the second term in the product can be interpreted as the sum of a geometric series with first term 1 and ratio $r = -x^3/3$, so

$$f(x) = \frac{1}{3}\sum_{n=0}^{\infty}\left(-\frac{x^3}{3}\right)^n = \frac{1}{3}\sum_{n=0}^{\infty}\frac{(-1)^n x^{3n}}{3^n} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{3n}}{3^{n+1}},$$

where in this last step we brought the 1/3 into the sum, giving us an extra factor of 3 in the denominator. This will converge if $|-x^3/3| < 1$, so $|x^3| < 3$, or $|x| < \sqrt[3]{3}$. So the radius of convergence is $\sqrt[3]{3}$.

Example 20.12. *Find a power series for $f(x) = \frac{x}{2+x}$ centered at 0. What is the radius of convergence?*

Solution. Again, the two in the denominator is an issue, so we factor it out:

$$f(x) = \frac{x}{2+x} = \frac{1}{2} \cdot \frac{x}{1+\frac{x}{2}}.$$

Now the second term in the product is the sum of a geometric seires with first term x and ratio $-x/2$, so

$$f(x) = \frac{1}{2}\sum_{n=0}^{\infty} x\left(-\frac{x}{2}\right)^n = \frac{1}{2}\sum_{n=0}^{\infty} x \cdot \frac{(-1)^n x^n}{2^n} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{2^{n+1}}.$$

This will converge if $|-x/2| < 1$, so $|x| < 2$. Therefore the radius is 2.

Example 20.13. *Find a power series for $f(x) = \frac{1}{2+x}$ centered at $x = 1$. What is the radius of convergence?*

Solution. Now the center is not 0, so we must be clever. In theory, it would be good if we had something like $\frac{1}{1+(x-1)}$, since then we could use the geometric series formula with $r = -(x-1)$, giving us a power series around 1. However, we don't have this. But, we can always manipulate the function so that we do have it:

$$f(x) = \frac{1}{2+x} = \frac{1}{2+x-1+1} = \frac{1}{3+(x-1)}.$$

Now we have a ratio involving $x-1$, so we're in business. Again, factor out the 3 to put a 1 in that spot:

$$f(x) = \frac{1}{3} \cdot \frac{1}{1+\frac{x-1}{3}}.$$

So now we have a geometric series with ratio $-\frac{x-1}{3}$ and first term 1, so

$$f(x) = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{-(x-1)}{3}\right)^n = \frac{1}{3}\sum_{n=0}^{\infty}\frac{(-1)^n(x-1)^n}{3^n} = \sum_{n=0}^{\infty}\frac{(-1)^n(x-1)^n}{3^{n+1}}.$$

This will converge if $\frac{x-1}{3} < 1$, so $|x-1| < 3$, and the radius of convergence is 3.

The last topic from this section is differentiation and integration of power series. What the theorem says is that this can be done term by term:

Theorem 20.14. *Suppose $F(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \ldots = \sum_{n=0}^{\infty} a_n(x-c)^n$ is a power series with radius of convergence R. Then*

$$F'(x) = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \ldots = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1},$$

and

$$\int F(x)dx = C + a_0(x-c) + \frac{a_1}{2}(x-c)^2 + \ldots = C + \sum_{n=0}^{\infty}\frac{a_n}{n+1}(x-c)^{n+1},$$

where C is any constant. Moreover, the radius of covergence is still R (though the interval of convergence may change).

Example 20.15. *Find a power series for $f(x) = \frac{2x}{x^2+1}$ around 0, and use it to find a power series for $F(x) = \ln(x^2+1)$ around 0. What is the radius of convergence for each?*

Solution. To find a power series for $f(x)$, we use the method above: this is the sum of a geometric series with first term $2x$ and ratio $-x^2$, so

$$f(x) = \sum_{n=0}^{\infty} 2x(-x^2)^n = \sum_{n=0}^{\infty} 2(-1)^n x^{2n+1}.$$

This will converge if $|-x^2| < 1$, so $|x|^2 < 1$, meaning $|x| < 1$, so the radius of convergence is 1 for this series. Next, we observe that $\ln(x^2+1)$ is an antiderivative of $\frac{2x}{x^2+1}$, i.e.

$$\int \frac{2x}{x^2+1} = \ln(x^2+1) + C.$$

You can check this using u-sub. So we can integrate the power series we got for $f(x)$ to get one for $F(x)$. Using the formula from the theorem, we get

$$F(x) = C + \sum_{n=0}^{\infty} \frac{2 \cdot (-1)^n}{2n+2} x^{2n+2}$$

(since the antiderivative of x^{2n+1} is $\frac{1}{2n+2}x^{2n+2}$). To solve for C, notice that if $x = 0$, then $F(0) = 0$. Plugging in $x = 0$ to both sides gives

$$0 = F(0) = C + \sum_{n=0}^{\infty} \frac{2(-1)^n}{2n+2} 0^{2n+2} = C,$$

so $C = 0$, and

$$F(x) = \sum_{n=0}^{\infty} \frac{2 \cdot (-1)^n}{2n+2} x^{2n+2}.$$

By the theorem, the radius of convergence stays 1 for this function as well.

Example 20.16. *Use the power series for $f(x) = \frac{1}{1-x}$ to get the power series for $\frac{1}{(1-x)^2}$, and then use it to find the power series for $\frac{1}{(1-x^2)^2}$. What is the radius of convergence for each?*

Solution. The power series for $f(x)$ is

$$f(x) = \sum_{n=0}^{\infty} x^n.$$

Observe that $\frac{1}{(1-x)^2}$ is the derivative of $\frac{1}{1-x}$, so differentiating the power series for $f(x)$ gives

$$f'(x) = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}.$$

Since the radius of convergence for the original series was 1, it remains 1 now. To get the last power series, we substitute x^2 everywhere we see an x in the series of $f'(x)$:

$$f'(x^2) = \frac{1}{(1-x^2)^2} = \sum_{n=1}^{\infty} n(x^2)^{n-1} = \sum_{n=1}^{\infty} n x^{2n-2}.$$

This will converge if $|x^2| < 1$, so $|x| < 1$, and the radius is still 1.